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School of Electrical Engineering
CORNELL UNIVERSITY
Ithaca, New York

RESEARCH REPORT EE 515

THEORETICAL AND EXPERIMENTAL INVESTIGATION OF LINEAR BEAM MICROWAVE TUBES

Final Report, Part II B

1 April 1957 to 31 August 1961

Approved by

G. Conrad Dalman
Lester F. Eastman
Paul R. McIsaac

Published under Contract No. AF30(602)-1696
Rome Air Development Center, Griffiss Air Force Base, N. Y.

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FINAL REPORT

PART II B:

BEAM-CIRCUIT INTERACTION

ON THE NONLINEAR THEORY OF THE PLANE KLYSTRON TUBE

Sven Olving

School of Electrical Engineering
CORNELL UNIVERSITY
Ithaca, New York

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ON THE NONLINEAR THEORY OF THE PLANE KLYSTRON TUBE

S. Olving

LINEAR BEAM MICROWAVE TUBES

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ABSTRACT

This note deals with the theory of an infinitely wide homogeneous electron stream which is velocity modulated in a plane perpendicular to the electron motion. The study is based on the exact wave equation and its exact solutions. The results are valid in the single velocity region only.

I. BASIC ASSUMPTIONS

The system we want to study consists of an infinitely wide homogeneous electron plasma stream, which is constrained to move in the positive z-direction. The composition of the plasma is such that the undisturbed d-c stream is electrically neutral. In the undisturbed case both the electrons and the ions move in a field-free space with the velocity v_o, while the corresponding convection current densities are i_o and $-i_o$. The presence of the neutralizing ions is necessary in order to satisfy Maxwell's third equation in the electric d-c field-free case. Furthermore, in the theoretical case of an infinitely wide stream, the transverse d-c magnetic field must vanish because the infinitely wide system is symmetrical with respect to any arbitrary longitudinal axis. It follows from Maxwell's first equation, which in the undisturbed case is identical to Ampere's law, that the transverse magnetic d-c field can vanish only when the total undisturbed direct-current density vanishes. This explains why we not only have to neutralize the undisturbed electronic d-c charge density but the undisturbed electronic d-c current density as well. In a transversely finite stream the ions can be assumed stationary but the edge effects on the propagation of the disturbances will then have to be taken into account, which leads to considerable complexity.[1]

The purpose of this study is to make an accurate investigation of how the system described propagates disturbances that may be produced by superimposing on the electrons a velocity modulation $v_{1o}(t)$, where t is time, at the plane $z = 0$. The positive ions are assumed to be so heavy that they are not affected by the disturbance.

The propagation of such disturbances takes place in the form of the so-called space-charge waves. In this case the waves will be plane and uniform. However, in dealing with space-charge waves, one usually linearizes the various equations, which means that cross products between the quantities describing the disturbances are neglected. In the present study no such approximations are made.

II. BASIC EQUATIONS

The purpose of this section is to formulate the basic equations describing the disturbed stream. We introduce the following notation:

$v(z, t) = v_o + v_1(z, t)$ = the electron velocity,

$\rho(z, t) = \rho_o + \rho_1(z, t)$ = the electron charge density,

$i(z, t) = i_o + i_1(z, t)$ = the electron current density,

$E(z, t) = E_o + E_1(z, t)$ = the (axial) electric field.

The quantities with subnotation "o" refer to the undisturbed system, while the quantities indexed by 1 represent the effects of the disturbance. According to our basic assumptions $E_o = 0$. Now, the position of an electron, z, can be written $z = z_o + z_1$ where z_o is the undisturbed position and z_1 the displacement from this position experienced by the electron because of the disturbance. Thus we have the following three relations:

$$v = \frac{dz}{dt} \qquad v_o = \frac{dz_o}{dt} \qquad v_1 = \frac{dz_1}{dt} \quad . \tag{1a, b, c}$$

According to the definition of convection current density, one has

$$i = \rho v \qquad i_o = \rho_o v_o \qquad i_1 = (\rho_o + \rho_1)\ (v_o + v_1) - \rho_o v_o \ . \tag{2a, b, c}$$

The equation of continuity,

$$\frac{\partial i_1}{\partial z} + \frac{\partial \rho_1}{\partial t} = 0 \quad , \tag{3}$$

is satisfied by introducing a quantity S in the following manner:

$$i_1 = \frac{\partial S}{\partial t} \qquad \rho_1 = -\frac{\partial S}{\partial z} \quad . \tag{4a, b}$$

By the use of Equations (4 a, b) and (1 a, b, c) in (2c), one obtains

$$\left(\frac{\partial}{\partial t} + \frac{dz}{dt}\frac{\partial}{\partial z}\right) S = \rho_o \frac{dz_1}{dt} .$$

The operator operating on S is just d/dt. Integration yields $S = \rho_o z_1$, which is used in Equations (4 a, b) to obtain

$$i_1 = \rho_o \frac{\partial z_1}{\partial t} \qquad \rho_1 = -\rho_o \frac{\partial z_1}{\partial z} . \tag{5 a, b}$$

The nonrelativistic equation of motion yields

$$m \frac{dz_1^2}{dt^2} = -eE_1 , \tag{6}$$

where $-e$ is the charge and m the mass of the electron. Since $\partial/\partial x = 0 = \partial/\partial y$ in the infinitely wide system, we get from Maxwell's first equation,

$$i_1 + \epsilon_o \frac{\partial E_1}{\partial t} = 0 , \tag{7}$$

where ϵ_o is the dielectric constant of free space. Equation (7) states that the total disturbance current density vanishes. By the use of Equations (5a) and (6) in Equation (7) one gets

$$\frac{\partial}{\partial t}\left(\rho_o z_1 - \frac{\epsilon_o m}{e} \frac{d^2 z_1}{dt^2}\right) = 0 ,$$

which is satisfied if

$$\frac{d^2 z_1}{dt^2} + \omega_p^2 z_1 = f(z) = 0 ,$$

where

$$\omega_p^2 = \frac{-e\rho_o}{m\epsilon_o} .$$

The function $f(z)$ arises from integration. However, it must equal zero since Equation (8) must hold even for the undisturbed case $z_1 = 0$.

Equation (8), of course, is valid only when the disturbance is such that z_1 is a single-valued function of z, that is, electron overtaking must not occur. If electrons with different displacements, say z_{11}, z_{12}, ... z_{1s}, simultaneously exist in a given plane z, then we must use the expression,

$$i_1 = \sum_{n=1}^{s} \rho_o \frac{\partial z_{1n}}{\partial t} \quad ,$$

rather than Equation (5a) to eliminate i_1 from Equation (7), and the problem becomes extremely complicated. Equation (8) has the general solution,

$$z_1 = A \sin \omega_p t + B \cos \omega_p t, \tag{9}$$

where A and B are constants. This shows that the electrons in any given z_o plane oscillate harmonically with the (plasma) frequency ω_p around the undisturbed position z_o ($=v_o t$) independently of the motion of other electrons, provided the given plane is not overtaken by other electron planes. It is extremely interesting to observe that the plasma frequency ω_p is the same as the one obtained in a linearized theory.

III. EXACT NONLINEAR PLANE KLYSTRON WAVES

The question arises as to where the nonlinearities are imbedded with respect to the fact that Equation (8) describes linear oscillations. The answer to this question is discussed in the present chapter.

By the use of Equation (1b) one can write Equation (8) in the form,

$$\frac{d^2 z_1}{dz_o^2} + \beta_p^2 z_1 = 0 \quad , \tag{10}$$

where

$$\beta_p = \omega_p / v_o \quad .$$

The general solution to Equation (10) is

$$z_1 = F_1 \sin \beta_p z_o + F_2 \cos \beta_p z_o \quad ; \tag{11}$$

$F_{1,2}$ are constants, that is, they must satisfy the equation,

$$\frac{dF_{1,2}}{dz_o} = 0 \quad . \tag{12}$$

Now, the expression $t - \frac{z_o}{v_o}$ is constant for any given electron, since

$$\frac{d\left(t - \frac{z_o}{v_o}\right)}{dz_o} = 0 \quad , \tag{13}$$

which one easily proves by the use of Equation (1b). This means that $F_{1,2}$ can be considered as two arbitrary functions of $t - \frac{z_o}{v_o}$. By doing so we account for the fact that although $F_{1,2}$ are constants for a given electron, the constants associated with another electron in another plane are, in general, different. Since an electron (plane) is identified by its characteristic constant, $t - \frac{z_o}{v_o}$, it follows that the amplitude factors

$F_{1,2}$ must be functions of this characteristic constant. Observe that $t - \frac{z_o}{v_o} = t_1$ where t_1 is the time when the electron plane was located at $z_o = 0$. Thus one can write

$$z_1 = F_1\left(t - \frac{z_o}{v_o}\right) \sin \beta_p z_o + F_2\left(t - \frac{z_o}{v_o}\right) \cos \beta_p z_o \quad , \qquad (14)$$

and, with Equations (1b) and (12),

$$v_1 = \frac{dz_1}{dt} = \beta_p v_o \left[F_1\left(t - \frac{z_o}{v_o}\right) \cos \beta_p z_o - F_2\left(t - \frac{z_o}{v_o}\right) \sin \beta_p z_o\right]. \qquad (15)$$

It is clear that one wishes to express the disturbance quantities in terms of the actual distance co-ordinate z rather than in terms of the undisturbed position co-ordinate z_o. For instance, if one wants to calculate i_1 from Equation (14) by the use of Equation (5a), which contains a time derivative of z_1 for a fixed distance z, one has to eliminate z_o in the right-hand side of Expression (14) by the use of the relation $z_o = z - z_1$. Equation (14) now becomes

$$z_1 = F_1\left(t - \frac{z - z_1}{v_o}\right) \sin \beta_p(z - z_1) + F_2\left(t - \frac{z - z_1}{v_o}\right) \cos \beta_p(z - z_1) \quad . \qquad (16)$$

This result represents a transcendental equation for z_1 rather than an explicit expression. Now, F_1 and F_2 have to be determined from the boundary values at the source of disturbance located at, say, $z = 0$. It is clear from Equation (16) that $z_1(t, z)$ is a nonlinear function of the boundary values, and the phenomena therefore are nonlinear when observed from a fixed point in space.

Suppose, for the sake of simplicity, that the electron stream is velocity modulated in an infinite gridded gap of zero length at the plane $z = 0$. Thus $z_1 = 0$ at $z = 0$, which imposes $z_o = 0$. From Equation (14) one now concludes that $F_2 = 0$. Furthermore, if we assume that the electrons leave the gap with a velocity $v_o + v_{1o}(t)$, where $v_{1o}(t)$ is not necessarily a periodic disturbance, we obtain from Equation (15)

$$\beta_p v_o F_1(t) = v_{1o}(t) \quad .$$

Equation (14) can now be written

$$z_1 = \frac{v_{1o}\left(t - \dfrac{z_o}{v_o}\right)}{v_o} \frac{1}{\beta_p} \sin \beta_p z_o \quad , \tag{17}$$

whereas Equation (15) yields

$$v_1 = v_{1o} \left(t - \frac{z_o}{v_o}\right) \cos \beta_p z_o \quad . \tag{18}$$

Upon elimination of z_o between Equation (17) and $z = z_o + z_1$, we obtain

$$z_1 = \frac{v_{1o}\left(t - \dfrac{z-z_1}{v_o}\right)}{v_o} \frac{1}{\beta_p} \sin \beta_p (z-z_1) \quad . \tag{19}$$

From Equation (19) one can express $\partial z_1/\partial t$ and $\partial z_1/\partial z$. By the use of (5a, b) we now obtain

$$\frac{i_1}{i_o} = \frac{\dfrac{v_{1o}'}{v_o^2 \beta_p} \sin \beta_p z_o}{1 + \dfrac{v_{1o}}{v_o} \cos \beta_p z_o - \dfrac{v_{1o}'}{v_o^2 \beta_p} \sin \beta_p z_o} \tag{20}$$

and

$$\frac{\rho_1}{\rho_o} = \frac{\dfrac{v_{1o}'}{v_o^2 \beta_p} \sin \beta_p z_o - \dfrac{v_{1o}}{v_o} \cos \beta_p z_o}{1 + \dfrac{v_{1o}}{v_o} \cos \beta_p z_o - \dfrac{v_{1o}'}{v_o^2 \beta_p} \sin \beta_p z_o} \quad , \tag{21}$$

where

$$v_{1o} = v_{1o}\left(t - \frac{z_o}{v_o}\right) \quad \text{and } v_{1o}' = \frac{dv_{1o}\left(t - \frac{z_o}{v_o}\right)}{d\left(t - \frac{z_o}{v_o}\right)} \quad .$$

Since, from Equations (6) and (8), $E_1 = \frac{m}{e}\ \omega_p^2 z_1$, we finally get, with Equation (17),

$$E_1 = \frac{m}{e}\ v_o \beta_p v_{1o} \sin \beta_p z_o \quad . \tag{22}$$

In principle one can now plot any one of the disturbance quantities v_1, i_1, ρ_1 and $E_1(\propto z_1)$ as functions of z_o for any fixed time t, provided the modulation function $v_{1o}(t)$ is given. Since z_1, too, can be plotted versus z_o, it becomes possible to plot v_1, i_1, ρ_1 and E_1 versus z ($=z_o+z_1$), the actual distance co-ordinate. The practical value of such graphs is limited, however. Instead one would like to have, for example, i_1 expressed in a Fourier series in time. We deal with this question in the next chapter.

Finally we formulate the condition that assures that no overtaking occurs. Suppose we plot z ($=z_o + z_1$) by the use of Equation (17) as a function of z_o for a fixed time. One immediately infers that electron overtaking has occurred in regions where the slope of our curve is negative. Thus $\partial z/\partial z_o > 0$, or,

$$\frac{\partial z_1}{\partial z_o} > -1 \quad , \tag{23}$$

which is the same condition we are looking for.

IV. D-C "MODULATED" STREAM

In order to point out a peculiar property of the infinitely wide stream, we will assume that the stream is "modulated" by a d-c velocity step. Thus the electron velocity is $v_o + v_1^o$, where $|v_1^o| =$ constant $< v_o$, at $z = 0$, whereas the ion velocity is unchanged. From Equations (17) and (18) one now obtains

$$z_1 = \frac{v_1^o}{v_o} \frac{1}{\beta_p} \sin \beta_p z_o \tag{24}$$

and

$$v_1 = v_1^o \cos \beta_p z_o \quad , \tag{25}$$

rather than

$$z_1 = \frac{v_1^o}{v_o} z_o \tag{26}$$

and

$$v_1 = v_1^o \quad , \tag{27}$$

which are the results one would obtain in a practical system with finite dimensions. However, it has already been pointed out in Section I that because of Maxwell's third equation, an infinitely extended d-c stream of electrons cannot exist unless the charged densities resulting from positive ions and electrons neutralize. The charge density of electrons obviously is $i_o/(v_o + v_1^o) \neq \rho_o$ in the d-c beam described by Equations (26) and (27). These Equations are therefore not related to the infinite beam.

Equations (24) and (25) are our exact solutions. They demonstrate that the electrons, as in the case of a-c modulation (Section V), oscillate around the undisturbed position z_o. It is clear that the average velocity of each electron must equal v_o, that is, the electron must keep in step with the positive ions, if we are to avoid an electrostatic catastrophe. This is true regardless of the type of modulation impressed on the electrons.

Those acquainted with the linearised space-charge-wave theory of the radially finite beam, recall that if the beam radius is large, then the plasma frequency reduction factors for sufficiently high signal frequencies approach unity.[1] At zero signal frequency, the reduction factor is zero, which corresponds to Equations (26) and (27).

If we linearize Equations (24) and (25), that, is if we write

$$z_1 = \frac{v_1^o}{v_o} \frac{1}{\beta_p} \sin \beta_p z \quad ,$$

$$v_1 = v_1^o \cos \beta_p z \quad ,$$

we immediately see that the reduction factor for the theoretical infinite stream at zero signal frequency is unity and not zero, as it is for a very large but finite beam.

It is now clear that the infinitely wide electron stream necessitates assumptions concerning the ions(Section I), imposing certain restrictions on the electron dynamics that are not present in practical devices, in addition to the fact that edge effects are neglected.

V. SINUSOIDALLY MODULATED STREAM

Let us assume for simplicity that the velocity modulation function is purely sinusoidal, that is,

$$v_{1o}(t) = v_1^o \quad \sin \omega t \quad , \tag{28}$$

where v_1^o is constant. This modulation requires a voltage, $V_1(t)$, across the infinitely short modulating gap given by

$$\frac{V_1(t)}{2V_o} = \frac{v_1^o}{v_o} \quad \sin \omega t + \frac{1}{2} \left(\frac{v_1^o}{v_o}\right)^2 \quad \sin^2 \omega t \quad , \tag{29}$$

where $V_o \left(= \frac{mv_o^2}{2e}\right)$ is the undisturbed beam voltage. We shall later deal with the more practical case of a purely sinusoidal gap voltage. Equation (17) now becomes

$$z_1 = \frac{v_1^o}{v_o} \quad \frac{1}{\beta_p} \quad \sin \beta_p z_o \quad \sin (\omega t - \beta_e z_o) \quad . \tag{30}$$

The validity of Equation (30) is, of course, limited by Condition (23), which yields

$$\frac{\partial z_1(z_o, t)}{\partial z_o} = \frac{v_1^o}{v_o \beta_p} (\beta_p \cos \beta_p z_o \sin T - \beta_e \sin \beta_p z_o \cos T) > -1 \ , \tag{31}$$

where $T = \omega t - \beta_e z_o$. Furthermore, we can determine T so that $\partial z_1/\partial z_o$ is maximally negative from the equation,

$$\frac{\partial z_1^2(z_o, t)}{\partial z_o \partial t} = 0 \quad , \tag{32}$$

which yields

$$\cot T + \frac{\beta_e}{\beta_p} \quad \tan \beta_p z_o = 0 \quad . \tag{33}$$

By the use of Equation (33) in Equation (31), one finds that overtaking does not occur during any part of the cycle, if

$$\sin^2\beta_p z_o \mathrel{\overline{<}} \frac{\left(\frac{v_o}{v_1^o}\right)^2 - 1}{\left(\frac{\beta_e}{\beta_p}\right)^2 - 1} \quad . \tag{34}$$

If we consider the expression (34) as an equality, the real solution $\beta_p z_o$, if it exists, is the $\beta_p z_o$ value at which overtaking first occurs. If the solution is imaginary, overtaking does not occur. It should be pointed out that Expression (34) is of interest only when $\beta_p \mathrel{\overline{<}} \beta_e$. If $\beta_p > \beta_e$, overtaking does not occur at all unless $|v_1^o| > v_o$, which we exclude, of course.

By the use of Equations (33) and (34) in Equation (30), we can express the displacement when overtaking is just about to take place, as

$$\beta_p z_1 = - \frac{v_1^o \beta_p}{v_o \beta_e} \frac{\sqrt{\left[1 - \left(\frac{v_1^o}{v_o}\right)^2\right]\left[1 - \left(\frac{v_o \beta_p}{v_1^o \beta_e}\right)^2\right]}}{1 - \left(\frac{\beta_p}{\beta_e}\right)^2} \quad . \tag{35}$$

If $(v_1^o)^2 \ll v_o^2$ and $\beta_p^2 \ll \beta_e^2$; then $\beta_p z_1 \to 0$ according to Equation (35), and Expression (34) becomes

$$\sin\beta_p z \mathrel{\overline{<}} \frac{v_o \beta_p}{v_1^o \beta_e} \; \frac{1}{M} \quad , \tag{36}$$

where

$$M = \sqrt{\frac{1 - \left(\frac{\beta_p}{\beta_e}\right)}{1 - \frac{v_1^o}{v_o}}} \quad .$$

Thus, when $M < 1$, overtaking does not occur and when $M = 1$ overtaking occurs at a distance $\frac{1}{4}\lambda_p = \left(\pi/2\beta_p\right)$, where λ_p is the plasma wavelength. If $M > 1$, overtaking occurs at a distance $z < \frac{1}{4}\lambda_p$ given by $M \sin\beta_p z = 1$. The distance is measured from the modulating gap.

In the ballistic klystron theory of Webster,[2] electron overtaking occurs at a distance given by $X = 1$, where $X = \frac{v_1^o}{v_o} \beta_e z$ is the bunching parameter. It is easy to see that in the present theory the overtaking distance is given by $X_p = X \frac{\sin\beta_p z}{\beta_p z} = 1$.

We will now attempt to express z_1 explicitly in the form of a Fourier series. If $|v_1^o| \ll v_o$, one finds from Equation (30) that $|\beta_p z_1| \ll 1$. Thus Equation (30) may be written approximately as

$$z_1 = \frac{v_1^o}{v_o} \frac{1}{\beta_p} \sin\beta_p z \sin(\omega t - \beta_e z_o) \quad . \tag{37}$$

With the notations $\omega t - \beta_e z_o = \theta$ and $\omega t - \beta_e z = \theta_o$, we can rewrite Equation (37) in the form,

$$\theta - \theta_o = X_p \sin\theta \quad , \tag{38}$$

which may be expanded in a Fourier series

$$\theta - \theta_o = \sum_{n=-\infty}^{n=\infty} D_n e^{jn\theta_o} \quad , \tag{39}$$

where

$$D_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\theta - \theta_o)\, e^{-jn\theta_o}\, d\theta_o \quad . \tag{40}$$

If n = 0 one obtains

$$D_o = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\theta - \theta_o) \cdot \frac{d\theta_o}{d\theta}\, d\theta \quad . \tag{41}$$

By the use of Equation (38) one obtains $D_o = 0$.

If $n \neq 0$ Equation (40) yields

$$D_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\theta - \theta_o)\, e^{-jn\theta_o}\, d\theta_o \quad ,$$

or

$$D_n = \frac{1}{2\pi} \Big|_{\theta_o=-\pi}^{\pi} (\theta-\theta_o) \frac{1}{(-jn)}\, e^{-jn\theta_o} - \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{d(\theta-\theta_o)}{d\theta_o} \frac{1}{(-jn)}\, e^{-jn\theta_o}\, d\theta_o \, ,$$

where the first term is equal to zero. Therefore

$$D_n = \frac{1}{j2\pi n} \int_{-\pi}^{\pi} e^{-jn\theta_o}\, d\theta \quad . \tag{42}$$

Use has been made of the fact that Equation (38) can also be written as $(\theta + 2\pi) - (\theta_o + 2\pi) = X_p \sin(\theta + 2\pi)$. Upon elimination of θ_o between Equations (42) and (38), one obtains

$$D_n = \frac{1}{j2\pi n} \int_{-\pi}^{\pi} e^{j(nX_p \sin\theta - n\theta)}\, d\theta = \frac{1}{jn}\, J_n(nX_p) \quad , \tag{43}$$

Remembering the integral representation for the Bessel functions of the first kind, we get

$$J_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j(x \sin \gamma - n\gamma)}\, d\gamma \quad .$$

We finally insert Equation (43) into Equation (39) to obtain

$$\theta - \theta_o = \sum_{\substack{n=-\infty \\ n \neq 0}}^{n=\infty} \frac{1}{jn} J_n(nX_p)\, e^{jn\theta_o} \quad . \tag{44}$$

Since $J_{-n}(-nX_p) = J_n(nX_p)$, we can rewrite Equation (44) in the form

$$z_1 = \frac{2}{\beta_e} \sum_{n=1}^{n=\infty} \frac{1}{n} J_n(nX_p) \sin n(\omega t - \beta_e z) \quad . \tag{45}$$

For small X_p one obtains,

$$z_1 \simeq \frac{v_1^o}{v_o} \frac{1}{\beta_p} \sin \beta_p z \sin(\omega t - \beta_e z) + \frac{1}{2}\left(\frac{v_1^o}{v_o}\right)^2 \frac{\beta_e}{\beta_p^2} \sin^2 \beta_p z \sin 2(\omega t - \beta_e z) \quad . \tag{46}$$

The normalized a-c current density can be expressed in the form,

$$\frac{i_1}{i_o} = \frac{\rho_o \frac{\partial z_1}{\partial t}}{\rho_o v_o} = 2 \sum_{n=1}^{n=\infty} J_n(nX_p) \cos n(\omega t - \beta_e z) \quad . \tag{47}$$

Equation (46) is formally identical to the corresponding result of the classical klystron theory of Webster.[3] The difference lies in the fact that the bunching parameter, X_p, takes the effect of the space-charge debunching forces into account whereas the classical bunching parameter, X, does not.

It is well known that $X_p = 1.84$ corresponds to maximum a-c current density (saturation) at the fundamental frequency (n=1). The maximum efficiency is then 58 per cent. However, $X_p = 1.84$ does not obey our nonovertaking condition, $X_p < 1$, which means that the theory is not valid in the range of maximum efficiency. This fact is fully in accord with our expectations. Suppose we observe the a-c current density at a fixed distance $z \gtrless \lambda_p/4$. Also assume, to begin with, that the drive is such that overtaking does not occur, that is, $X_p < 1$. One easily infers now that an increase in drive (v_1^o) will increase the fundamental frequency a-c current since all electrons will be brought closer to the center of the bunch. However, if we increase the drive until electron overtaking occurs at the observation point, then some electrons overtake the center of the bunch and move away from it. These electrons tend to weaken the bunch, whereas others still move towards the center. It is clear that an optimum drive level exists for each distance z, the saturation drive, such that a further increase in drive would lead to a weaker fundamental frequency a-c current because of the effect of electron overtaking. Obviously one can increase the maximum efficiency of a klystron by delaying overtaking. We shall deal with this question in the next section.

Recently Paschke[4] has studied the planar klystron by solving the partial differential equation of the problem by the use of the successive approximation method. His results are obtained in the form of expansions of the type in Equation (44) rather than in the form of the more accurate series of Equation (45). According to Paschke's theory, saturation occurs for $X_p = \sqrt{8/3}$. Unfortunately Paschke has not pointed out that this X_p

value is well within the overtaking range, where the theory is no longer valid. Although $X_p = \sqrt{8/3}$ (or our result $X_p = 1.84$) is certainly close to the actual saturation value, one can hardly deal confidently with problems related to saturation unless one properly includes the phenomenon that produces saturation – electron overtaking – in the theory. An approximate theory of this type for the infinitely wide stream has been developed by Roe in unpublished notes. Some of Roe's results have been discussed by Mihran.[5]

Next we want to study the case of purely sinusoidal voltage modulation $V_1^o \sin\omega t$. From the energy relation in the modulation gap, one obtains

$$1 + \left(\frac{v_{1o}(t)}{v_o}\right)^2 = 1 + \frac{V_1^o}{V_o} \sin\omega t \quad , \tag{48}$$

or to the second-order accuracy,

$$\frac{v_1^o(t)}{v_o} \simeq \frac{1}{2}\frac{V_1^o}{V_o} \sin\omega t - \frac{1}{8}\left(\frac{V_1^o}{V_o}\right)^2 \sin^2\omega t \quad . \tag{49}$$

If $\beta_p z_1 \ll 1$, one gets from Equation (17) and $z = z_o + z_1$, to the second-order accuracy,

$$z_1 \simeq \frac{v_{1o}\left(t - \frac{z_o}{v_o}\right)}{v_o\,\beta_p} \left(\sin\beta_p z - \beta_p z_1 \cos\beta_p z\right) \quad , \tag{50}$$

or with Equation (49),

$$z_1 \simeq \frac{1}{2}\frac{V_1^o}{V_o}\frac{1}{\beta_p} \sin\beta_p z \sin(\omega t - \beta_e z_o)$$

$$- \frac{1}{8}\left(\frac{V_1^o}{V_o}\right)^2 \frac{1}{\beta_p} \left(\sin\beta_p z + \sin 2\beta_p z\right) \sin^2(\omega t - \beta_e z_o) \quad . \tag{51}$$

With the notations

$$X_{p1} = \frac{1}{2} \frac{V_1^o}{V_o} \frac{\beta_e}{\beta_p} \sin \beta_p z \quad ,$$

and

$$X_{p2} = \frac{1}{8} \left(\frac{V_1^o}{V_o}\right)^2 \frac{\beta_e}{\beta_p} (\sin \beta_p z + \sin 2\beta_p z) \quad ,$$

one can rewrite Equation (52) in the form,

$$\theta - \theta_o = X_{p1} \sin \theta - X_{p2} \sin^2 \theta \quad . \tag{52}$$

If we expand Equation (52) in a Fourier series by the use of Equations (39) and (40), we will obtain

$$D_o = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\theta - \theta_o) \frac{d\theta_o}{d\theta} d\theta = -\frac{X_2}{2\pi} \int_{-\pi}^{\pi} \sin^2 \theta \, d\theta = -\frac{X_{p2}}{2} \quad , \tag{53}$$

while

$$D_{n_{n \neq o}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\theta - \theta_o) e^{-jn\theta_o} d\theta_o = \frac{1}{j2\pi n} \int_{-\pi}^{\pi} e^{-jn\theta_o} d\theta =$$

$$\frac{1}{j2\pi n} \int_{-\pi}^{\pi} e^{jn(X_{p1} \sin \theta - X_{p2} \sin^2 \theta - \theta)} d\theta$$

or with

$$e^{-jnX_{p2} \sin^2 \theta} \simeq 1 - jn X_{p2} \sin^2 \theta$$

$$D_{n_{n \neq o}} = \frac{1}{jn} J_n(nX_{p1}) - \frac{X_{p2}}{2\pi} \int_{-\pi}^{\pi} e^{jn(X_{p1} \sin \theta - \theta)} \sin^2 \theta \, d\theta \quad . \tag{54}$$

Now, the integral in Equation (55) is of the order $2\pi J_n(nX_{p1})$. Thus the second term in Equation (55) is of the order $|n X_{p2}|$ times less than the first term. Since the approximation given by Equation (54) is based on the assumption $|nX_{p2}| \ll 1$, we see that under this condition the second-term in Equation (55) is not a very important one. Observe that nX_{p2} is of the order $\frac{\beta_p}{\beta_e} X_{p1}^2 \ll 1$ because we normally have $\beta_p \ll \beta_e$ in addition to $X_{p1} < 1$, which is the range of validity of the present theory.

By comparing Equations (55) and (43) we reach the conclusion that the effect of the nonlinearity in the initial condition [Expression (49)] is considerably less than that of the nonlinearities in the drift space. We should, however, make a comment concerning the fact that $D_o \neq 0$. From the relation $E_1 = \frac{m}{e} \omega_p^2 z_1$ and Equation (53), we find that there is a d-c electric field,

$$E_{1_{dc}} = -\frac{m}{e}\omega_p^2 \frac{1}{16}\left(\frac{V_1^{\sigma}}{V_o}\right)^2 \frac{\beta_e}{\beta_p} (\sin \beta_p z + \sin 2 \beta_p z) \quad , \qquad (55)$$

in the stream. The d-c potential disturbance becomes

$$V_{1_{dc}}(z) = \int_0^z -E_{1_{dc}} dz = \frac{m}{e}\omega v_o \frac{1}{16} \left(\frac{V_1^{\sigma}}{V_o}\right)^2 \left(\cos \beta_p z + \frac{1}{2} \cos 2\beta_p z - \frac{3}{2}\right) \quad . \quad (56)$$

Equation (56) shows that the d-c potential in the modulated free stream is in general not equal to V_o if the nonlinear effects are taken into account. This implies that an electrode, for instance a demodulating gap, with a d-c potential V_o at a distance z, would in general disturb the stream even at distances less than z, so that $V_{1_{dc}}(z)$ is made to vanish at the demodulating gap. We see from Equation (56) that only when z equals an integral number of plasma wavelengths does the potential disturbance automatically vanish.

VI. SAWTOOTH-MODULATED STREAM

The main object of this section is to find a periodic modulation function $v_{1o}(t)$ such that the electrons associated with any given period will arrive simultaneously at a plane $z = d < \frac{\lambda_p}{4}$ in the form of infinitely short bunches with infinite current density. Between the bunches the current density is zero, while the average current density is i_o. The advantage of such a modulation scheme is that one can convert practically all the kinetic energy of the electrons into a-c electromagnetic energy at the modulation frequency ω or at any one of the harmonic frequencies $n\omega$ $(n = 1, 2, 3, \ldots)$ by the use of a demodulating gap at $z = d$ tuned to the proper frequency.

Let us consider the modulation period $-\pi/\omega < t < \pi/\omega$. Suppose that $v_{1o}(0) = 0$. The problem is to determine the velocity modulation function $v_{1o}(t)$ in the given interval from the condition that the electrons leaving the modulating gap during that interval must simultaneously reach the distance d at a time d/v_o. Now, for a given electron, s, one obtains, substituting d/v_o for t in Equation (17),

$$z_{1s} = d - c_{os} = \frac{v_{1o}\left(\frac{d}{v_o} - \frac{z_{os}}{v_o}\right)}{v_o \beta_p} \sin \beta_p z_{os} \quad . \tag{57}$$

The expression $\frac{d}{v_o} - \frac{z_{os}}{v_o}$ is nothing else than the time, t, when the electron, s, left the modulating gap. Thus Equation (57) can be rewritten in the form,

$$\frac{v_{1o}(t)}{v_o} = \frac{\beta_p}{\beta_e} \quad \frac{\omega t}{\sin\left[\frac{\beta_p}{\beta_e}(\beta_e d - \omega t)\right]} \quad , \tag{58}$$

which expresses the periodic velocity-modulation function we have been looking for in the interval $-\pi/\omega < t < \pi/\omega$. If $\beta_p \ll \beta_e$ we can write Equation (58) in the approximate form,

$$\frac{v_{1o}(t)}{v_o} \simeq \frac{\beta_p}{\beta_e} \; \frac{\omega t}{\sin \beta_p d} \quad , \tag{59}$$

which is a sawtooth wave. If the velocity distribution in the bunch is small, that is if $v_{1o}(t) \ll v_o$, the maximum conversion efficiency will be approximately 100 per cent.

The Fourier expansion of the sawtooth modulation [Equation(59)] is

$$\frac{v_{1o}(t)}{v_o} = \frac{2\beta_p}{\beta_e \sin \beta_p d} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin n\omega t \quad . \tag{60}$$

The frequency spectrum of this modulation signal is infinite. It therefore cannot be conveniently produced if the fundamental frequency is in the microwave range. However, it would be feasible to produce a signal consisting of the first two harmonics in Equation (60), i. e.,

$$\frac{v_{1o}(t)}{v_o} = \frac{2\beta_p}{\beta_e \sin \beta_p d} \left(\sin \omega t - \frac{1}{2} \sin \omega t\right) \quad . \tag{61}$$

If we now calculate the fundamental frequency a-c current density amplitude at a distance d from the modulating gap, the result would be 1.5 i_o. The maximum fundamental frequency a-c current density in the purely sinusoidally modulated stream was 1.16 i_o according to Equation (47). The maximum efficiencies in the two cases are 75 per cent and 58 per cent respectively. Both figures are obtained by extrapolating the theory into the range past crossover. Significantly, however, these results indicate that the presence of the proper second harmonic in the modulation signal substantially improves the efficiency. Since a high-power klystron produces a considerable amount of harmonic frequency electro-magnetic power in the output gap, it would be conceivable to make use of this power in the input gap in order to approach the desirable sawtooth type modulation signal.

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NONLINEAR SPACE-CHARGE-WAVE THEORY OF THE RADIALLY FINITE ELECTRON BEAM

Sven Olving

School of Electrical Engineering
CORNELL UNIVERSITY
Ithaca, New York

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S. Olving

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ABSTRACT

The exact nonlinear wave equation for longitudinal space-charge waves in a well-confined cylindrical electron beam is derived under the assumption of no electron overtaking. A procedure is developed, based on the method of successive approximations, which makes it possible to find the solution in the form of Fourier-Bessel expansions satisfying the proper boundary conditions. In the present analysis the second-order solution is obtained without any restrictions concerning the diameter of the electron beam or the enclosing drift tube. The nonlinear part of the solution describes the properties of the second-harmonic frequency waves. In general the second harmonic contains linearly (with distance) growing terms. The growth of the second harmonic has been experimentally demonstrated by Mihran.[1] The results reduce to those of Paschke[2] in the case of very thin beams.

INTRODUCTION

The interest in the nonlinear behavior of electron beams has become considerable in recent years. Many electron beam devices, such as oscillators, are inherently nonlinear. Other devices, e.g., power amplifiers, are usually driven at such high signal levels that linear differential equations do not predict their behavior very accurately. Naturally it is desirable to develop theories for existing devices with the nonlinearities taken into account. Furthermore, detailed theoretical studies of nonlinear electron beams and plasmas may reveal phenomena upon which entirely new devices could be based.

The present report is concerned with the nonlinear aspects of the Hahn-Ramo waves[3,4] (space-charge waves) which have been used so successfully in the development of the microwave art. The differential equation for the Hahn-Ramo waves is usually obtained in the linear form, which means that cross products between the wave quantities are ignored in the various expressions. The solutions have been limited to low levels consistent with the linearizing approximations. The exact wave equation, however, is always nonlinear.

For infinitely thick beams the radial variations (and the radial boundary conditions) vanish, and the oscillatory properties of the electrons are entirely described by the wave equation. The nonlinear infinitely wide beam is therefore a comparatively simple problem.[5,6] However, the radial boundary conditions are of paramount importance in connection with radially finite beams and have to be taken into account

II. THE NONLINEAR SPACE-CHARGE-WAVE EQUATION FOR A RADIALLY FINITE BEAM

Consider a circular cylindrical electron beam propagating in the positive z-direction. The electron motion is restricted to the z-direction only, which can be achieved by applying a sufficiently strong d-c magnetic field. The undisturbed electron beam is axially homogeneous, i.e., the velocity, v_o, the electronic charge density, ρ_o, and the current density, $i_o (= \rho_o v_o)$, are independent of z. However, v_o, ρ_o, and i_o may vary with r, the radial co-ordinate of the circular cylindrical co-ordinate system (r, ϕ, z). No azimuthal variation will be allowed, i.e., $\partial/\partial\phi = 0$. The undisturbed beam is supposed to be electrically neutral, i.e., the ionic charge density is $-\rho_o$. The ions are assumed to be so heavy that they are not affected by a disturbance.

We wish to formulate the wave equation that, together with the boundary conditions, governs the propagation of azimuthally symmetrical disturbances along the electron beam. Contrary to the usual linearizing procedure, we will not neglect any cross products between the quantities describing the disturbance.

Denoting the electronic current density in the presence of the disturbance as $i_o(r) + i_1(r, z, t)$, where t is time, and remembering that $\partial/\partial\phi = 0$, we obtain from Maxwell's equations for the transverse-magnetic waves

$$\frac{1}{r}\frac{\partial}{\partial r}(rH_\phi) = i_1 + \epsilon_o \frac{\partial E_z}{\partial t} \quad , \tag{1}$$

$$-\frac{\partial H_\phi}{\partial z} = \epsilon_o \frac{\partial E_r}{\partial t} \quad , \tag{2}$$

$$\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} = -\mu_o \frac{\partial H_\phi}{\partial t} \quad , \tag{3}$$

where $E(r, z, t)$ and $H(r, z, t)$ denote the electric and magnetic fields respectively, while ϵ_o is the permittivity and μ_o the permeability of free space.

From Equations (2) and (3) one obtains

$$\epsilon_o \frac{\partial^2 E_z}{\partial t \partial r} = \left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) H_\phi \quad , \tag{4}$$

where $c = 1/\sqrt{\mu_o \epsilon_o}$ is the velocity of light in free space. Equation (4) will be useful in connection with the boundary value studies. By the elimination of H_ϕ between Equations (1) and (4), one obtains

$$\frac{\partial}{\partial t}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_z}{\partial r}\right)\right] - \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2}\right)\left(\frac{i_1}{\epsilon_o} + \frac{\partial E_z}{\partial t}\right) = 0 \quad . \tag{5}$$

Ignoring relativistic effects, we can make use of Newton's equation of motion,

$$m\frac{d^2 z_1}{dt^2} = -e E_z \quad , \tag{6}$$

where z_1 is the displacement of the electron under the influence of the disturbance while m is the mass and $-e$ the charge of the electron.

It is practical to write the final wave equation in terms of the displacement z_1. Since E_z can be eliminated from Equation (5) by the use of Equation (6), it remains to express i_1 in terms of z_1.

The equation of continuity,

$$\frac{\partial i_1}{\partial z} + \frac{\partial \rho_1}{\partial t} = 0 \quad , \tag{7}$$

is satisfied by writing

$$i_1 = \frac{\partial S}{\partial t} \quad \text{and} \quad \rho_1 = \frac{\partial S}{\partial t} \quad ; \tag{8}$$

ρ_1 is the excess electron density resulting from the presence of the disturbance. The quantity S can be expressed in terms of z_1 by the use of the relation that defines i_1, namely,

$$i_1 = (\rho_o + \rho_1)(v_o + v_1) - \rho_o v_o \quad , \tag{9}$$

where $v_1 = dz_1/dt$ while $v_o + v_1 = dz/dt$. Using Equation (8) in (9), we obtain

$$\left(\frac{\partial}{\partial t} + \frac{dz}{dt} \frac{\partial}{\partial z} \right) S = \rho_o \frac{dz_1}{dt} \quad .$$

The operator operating on S is just d/dt. Integration yields $S = \rho_o z_1$, which is used in Equation (8) to obtain

$$i_1 = \rho_o \frac{\partial z_1}{\partial t} \tag{10}$$

and

$$\rho_1 = -\rho_o \frac{\partial z_1}{\partial z} \quad . \tag{11}$$

The elimination of E_z and i_1 from Equation (5) by the use of Equations (6) and (10) gives the desired nonlinear wave equation

$$\boxed{\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r}\frac{d^2 z_1}{dt^2} - \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial t^2}\right)\left(\frac{d^2 z_1}{dt^2} + \omega_p^2 z_1\right) = 0} \tag{12}$$

The quantity ω_p is the angular plasma frequency defined by

$$\omega_p^2(r) = -\frac{e\rho_o(r)}{m\epsilon_o} \quad . \tag{13}$$

Equation (12), of course, is valid only when the disturbance is such that z_1 is a single-valued function of z, that is, electron overtaking must not occur. If electrons with different displacements, say $z_{11}, z_{12}, \ldots, z_{1s}$, exist simultaneously in a given plane z, then we must use

$$i_1 = \sum_{n=1}^{s} \rho_o \frac{\partial z_{1n}}{\partial t}$$

rather than Equation (10) to eliminate i_1 from Equation (5), and the problem becomes extremely complicated.

Equation (12) is a nonlinear partial differential equation because the time derivative d/dt becomes nonlinear when expressed in terms of partial derivatives.

Making use of the formula,

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \left(v_o + \frac{dz_1}{dt}\right)\frac{\partial}{\partial z} \quad , \tag{14}$$

we can express the excess electron velocity v_1 as

$$v_1 = \frac{dz_1}{dt} = \left(1 - \frac{\partial z_1}{\partial z}\right)^{-1} \dot{z}_1 \quad , \tag{15}$$

while

$$\frac{d^2 z_1}{dt^2} = \left(1 - \frac{\partial z_1}{\partial z}\right)^{-1} \ddot{z}_1 + \left(1 - \frac{\partial z_1}{\partial z}\right)^{-2} \frac{\partial}{\partial z} (\dot{z}_1)^2 + \left(1 - \frac{\partial z_1}{\partial z}\right)^{-3} (\dot{z}_1)^2 \frac{\partial^2 z_1}{\partial z^2} \quad , \tag{16}$$

where

$$\dot{z}_1 \equiv \left(\frac{\partial}{\partial t} + v_o \frac{\partial}{\partial z}\right) z_1 \quad ,$$

and

$$\ddot{z}_1 \equiv \left(\frac{\partial^2}{\partial t^2} + 2v_o \frac{\partial^2}{\partial t \partial z} + v_o^2 \frac{\partial^2}{\partial z^2}\right) z_1 \quad .$$

Thus the dot denotes the linearized time derivative.

If one uses Equation (16) in Equation (12) to eliminate $d^2 z_1/dt^2$, the result is the exact nonlinear partial differential equation of our problem. It is, however, a hopeless task to find such exact solutions for the exact wave equation, which satisfy the boundary conditions. One is therefore forced to deal with some finite order of accuracy. We will limit ourselves to the second-order approximation.

The total electronic charge density can never become positive, thus $\rho_o + \rho_1 \leq 0$. By the use of Equation (11), one easily infers that $\partial z_1/\partial z \leq 1$. Assuming $\left|\partial z_1/\partial z\right| < 1$ we can write

$$\left(1 - \frac{\partial z_1}{\partial z}\right)^{-1} \simeq 1 + \frac{\partial z_1}{\partial z}$$

and obtain, omitting all terms of higher than the second-order,

$$\frac{d^2 z_1}{dt^2} \simeq \ddot{z}_1 + a(z_1) \quad , \tag{17}$$

where

$$a(z_1) \equiv \frac{\partial z_1}{\partial z} \ddot{z}_1 + \frac{\partial}{\partial z} (\dot{z}_1)^2 \tag{18}$$

denotes the second-order terms. Introducing the operators P and Q defined by

$$P \equiv \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} \quad ,$$

$$Q \equiv - \frac{1}{c^2} \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial_z^2} \right) \quad ,$$

and using relation (17) in Equation (12), we obtain

$$\boxed{(P + Q)\ddot{z}_1 + Q\omega_p^2 z_1 = -(P + Q)\, a(z_1)} \quad . \tag{19}$$

Equation (19) is the desired nonlinear wave equation of second-order accuracy. All linear terms appear on the left-hand side while the right-hand terms are nonlinear.

III. THE FIRST-ORDER (LINEAR) MULTIMODE SOLUTION

We will ignore the nonlinear part of Equation (19) in this chapter in order to obtain the linearized solution of Equation (19). We shall need this solution in the next chapter in order to deduce the second-order solution. The linear multimode solution is, of course, well known and the reader is referred to the literature for a more extensive treatment[8, 9] than the one given here.

From now on, we assume that the undisturbed beam parameters i_o, v_o, and ρ_o are constants; i.e., they do not vary with the radial coordinate. We denote the linearized displacement by x and write from Equation (19),

$$(P + Q)\ddot{x} + Q\omega_p^2 \, x = 0 \quad . \tag{20}$$

The exact solution of this equation is

$$x = \sum_{k=1}^{\infty} A_k J_o(T_k r) \sin(\omega t - \gamma_k z + \theta_k) \quad . \tag{21}$$

It has been assumed that the disturbance is sinusoidal with an angular frequency ω. The propagation constants γ_k, of which there exists an infinite set, are determined by the radial boundary conditions, while the amplitude coefficients A_k and the phase constants θ_k are given by the initial conditions. The radial propagation constants T_k are defined by

$$T_k = h_k \left[\frac{\beta_p^2}{(\gamma_k - \beta_e)^2} - 1 \right]^{1/2} , \qquad (22)$$

where

$$h_k = (\gamma_k^2 - k_o^2)^{1/2} ,$$

$$\beta_p = \omega_p / v_o ,$$

$$\beta_e = \omega / v_o ,$$

$$k_o = \omega / c .$$

Matching the radial wave impedance at the edge of the beam yields an equation[9] from which the various γ_k can be determined, viz.,

$$T_k b \, \frac{J_1(T_k b)}{J_o(T_k b)} = hb \, \frac{K_1(hb) + \frac{K_o(ha)}{I_o(ha)} I_1(hb)}{K_o(hb) - \frac{K_o(ha)}{I_o(ha)} I_o(hb)} , \qquad (23)$$

where b is the radius of the beam and a is the radius of the lossless drift tube. Equations (21) and (23) include the electromagnetic transverse-magnetic waves in which we are not interested. For the space-charge waves we usually have $|\gamma_k - \beta_e| \ll \beta_e$. This means that we can write $h \simeq \beta_e$ and the right-hand side of Equation (23) becomes a known function. We shall denote it by $F(a, b, \beta_e)$. Furthermore, the propagation constants

γ_k can be grouped in symmetrical pairs expressed by

$$\gamma_{k_{1,2}} \simeq \beta_e \pm R_k\beta_p \quad , \tag{24}$$

where $R_k (\leq 1)$ is the so-called plasma frequency reduction factor of mode k; R_1 is the largest reduction factor, R_2 the next largest, etc.

Expression (21) can now be rewritten in the approximate form,

$$x \simeq \sum_{k=1}^{\infty} A_k \sin(R_k\beta_p z + \theta_{1,k})\, J_o(T_k r) \sin(\omega t - \beta_e z + \theta_{2,k}) \, , \tag{25}$$

where

$$T_k \simeq \left(\beta_e \quad \frac{1}{R_k^2} - 1\right)^{1/2} \quad , \tag{26}$$

and

$$T_k b \, \frac{J_1(T_k b)}{J_o(T_k b)} \simeq F(a, b, \beta_e) \quad . \tag{27}$$

Extensive graphs for the solution R_1 of Equation (27) have been given by Branch and Mihran,[10] while an explicit expression for R_k, valid for all k, has been derived by Olving. [11]

We will now assume that the beam is excited by pure velocity modulation $v_1^o \sin \omega t$ at the plane $z = 0$. Thus the initial conditions at the plane $z = 0$ are $x = 0$ and $\dot{x} = v_1 = v_1^o \sin \omega t$. One easily finds that $\theta_{1k} = 0 = \theta_{2k}$. The determination of A_k requires the application of the Fourier-Bessel expansion.[12] The result is

$$x = \frac{v_1^o}{v_o} \, \frac{1}{\beta_p} \sin(\omega t - \beta_e z) \sum_{k=1}^{\infty} \frac{\epsilon_k}{R_k} \sin(R_k\beta_p z)\, J_o(T_k r) \quad , \tag{28}$$

where T_k is determined by Equation (27) while ϵ_k denotes

$$\epsilon_k = \frac{\dfrac{J_1(T_k b)}{T_k b/2}}{J_o^2(T_k b) + J_1^2(T_k b)} \quad . \tag{29}$$

The normalized first-order a-c current density becomes

$$\frac{i_{1x}}{i_o} = \frac{\rho_o \frac{\partial z}{\partial t}}{\rho_o v_o} = \frac{v_1^o \beta_e}{v_o \beta_p} \cos(\omega t - \beta_e z) \sum_{k=1}^{\infty} \frac{\epsilon_k}{R_k} \sin(R_k \beta_p z) \, J_o(T_k r) \quad . \tag{30}$$

The normalized total first-order a-c current in the beam is

$$\frac{I_{1x}}{I_o} = \frac{\int_o^b i_{1x} 2\pi r dr}{\int_o^b i_o 2\pi r dr} = \frac{v_1^o \beta_e}{v_o \beta_p} \cos(\omega t - \beta_e z) \sum_{k=1}^{\infty} \frac{\epsilon_k}{R_k} \frac{J_1(T_k b)}{T_k b/2} \sin(R_k \beta_p z) \quad . \tag{31}$$

It is worth while noting that the significance of higher-order modes is greater in Equations (28) and (30) than in Equation (31). This means that the axial variation of the total a-c current may be essentially described by the fundamental mode $k = 1$, while the description of the detailed dynamic state of the same beam may require the higher-order modes to be taken into account.

IV. THE SECOND-ORDER MULTIMODE SOLUTION

The purpose of this chapter is to find the second-order nonlinear correction term y to the first-order displacement, x. Thus we write

$$z_1 \simeq x + y \quad , \tag{32}$$

where x, according to Equation (28), is proportional to the modulation index v_1^o/v_o while y is supposed to be proportional to $(v_1^o/v_o)^2$. If one uses relation (32) and Equation (20) in Equation (19) and ignores terms of higher order than $(v_1^o/r_o)^2$, the following nonhomogeneous linear differential equation will result:

$$(P + Q)\ddot{y} + \omega_p^2 \, Qy = -(P+Q)\,a(x) \quad . \tag{33}$$

The problem is now to solve this equation with the initial and boundary conditions taken into account. The function $a(x)$ can be expressed by the use of Equations (18) and (28) after multiplying the series and after some trigonometric manipulations, as

$$a(x) = -\frac{1}{4}\left(\frac{v_1^o}{v_o}\right)^2 \beta_e v_o^2 \sin 2(\omega t - \beta_e z) \sum_{k,m}^{\infty} \Bigg\{ \epsilon_k \epsilon_m \frac{1}{R_m} J_o(T_k r) J_o(T_n r) \cdot \left[(2R_m - R_k)\cos(R_m - R_k)\beta_p z + (2R_m + R_k)\cos(R_m + R_k)\beta_p z \right] \Bigg\} . \tag{34}$$

The sum $\sum_{k,m}$ is to be taken for all combinations between $k = 1, 2, 3, \ldots, \infty$ and $m = 1, 2, 3, \ldots, \infty$. Thus to make this clear,

$$\sum_{k,m} L(k,m) = L(1,1) + L(1,2) + L(1,3) + \ldots + L(1,\infty)$$

$$+ L(2,1) + L(2,2) + L(2,3) + \ldots + L(2,\infty)$$

$$+ \text{-----------------------------------}$$

$$+ L(\infty,1) + L(\infty,2) + L(\infty,3) + \ldots + L(\infty,\infty) \quad .$$

It should be pointed out that some terms, which should formally appear in Equation (34), have been neglected on account of the condition $\beta_e \quad R_k\beta_p$ introduced already in Chapter III.

Now, both sides of Equation (32) contain the Bessel operator P, indicating that the equation is separable provided the quantity $a(x)$ can be expanded in terms of linear Bessel functions, say $J_o(D_n r)$, $n = 1, 2, 3, \ldots, \infty$, instead of the products $J_o(T_k r)\, J_o(T_m r)$. Let us assume that the constants D_n satisfy the equation,

$$D_n b \, \frac{J_1(D_n b)}{J_o(D_n b)} = H \quad , \tag{35}$$

where H is a real constant independent of n. Equation (35) allows us to make use of the Fourier-Bessel expansion[12]

$$J_o(T_k r) \;\; J_o(T_m r) = \sum_{n=1}^{\infty} \epsilon_n(k,m) \;\; J_o(D_n r) \quad , \tag{36}$$

$$\epsilon_n(k, m) = \frac{\int_0^1 J_o(T_k r)\, J_o(T_m r)\, J_o(D_n r)\, \frac{r}{b}\, d\, \frac{r}{b}}{\frac{1}{2}\left[J_o^2\,(D_n b) + J_1^2\,(D_n b)\right]} \quad . \tag{37}$$

By the use of Equation (36) we can rewrite Equation (34) in the form,

$$a(x) = -\frac{1}{4}\left(\frac{v_1^o}{v_o}\right)^2 \beta_e v_o^2 \sin(\omega t - \beta_e z) \sum_{n=1}^{\infty} \left\{ \sum_{k,m}^{\infty} \epsilon_n(k, m)\, \epsilon_k \epsilon_m \frac{1}{R_m} \cdot \left[(2R_m - R_k)\cos(R_m - R_k)\,\beta_p z + (2R_m + R_k)\cos(R_m + R_k)\,\beta_p z\right]\right\} J_o(D_n r) \quad . \tag{38}$$

Since $P J_o(D_n r) = -D_n^2 J_o(D_n r)$ and $Q a(x) \simeq -(2\beta_e)^2 a(x)$, it is now a trivial problem to find a solution to the nonhomogeneous Equation (33). Furthermore, one easily infers by the use of Equation (4) that each term ("mode") in the solution, characterized by its particular radial wave function $J_o(D_n r)$, independently satisfies the radial boundary conditions, provided

$$D_n b \; \frac{J_1(D_n b)}{J_o(D_n b)} = F(a, b, 2\beta_e) \quad , \tag{39}$$

where the function F is the expression on the right-hand side of Equation (23) with h replaced by $2\beta_e$. Observe, however, that the various $J_o(D_n r)$ "modes" taken separately are not solutions to the nonhomogeneous Equation (33), only the sum [Equation (41)] is a solution.

Since F is constant and independent of n, Equation (39) shows that condition (35), upon which the Fourier-Bessel expansion was based, is satisfied. Equation (39) is identical to the equation that determines the linear theory plasma-frequency reduction factors of the different modes n at the frequency 2ω. Thus we can write as in relation (26),

$$D_n = 2\beta_e \left(\frac{1}{S_n^2} - 1 \right)^{1/2} , \tag{40}$$

where S_n is the linear theory plasma-frequency reduction factor of mode n evaluated at the signal frequency 2ω.

With these notations the solution to the nonhomogeneous Equation (33) can be written

$$y_{nonhom} = \frac{1}{2} \left(\frac{v_1}{v_o} \right)^2 \frac{\beta_e}{\beta_p^2} \sin 2(\omega t - \beta_e z) \sum_{n=1}^{\infty} \left\{ \sum_{k,m} \epsilon_n(k,m) \epsilon_k \epsilon_m \frac{1}{R_m} \Psi_n(k,m,z) \right\} J_o(D_n r) , \tag{41}$$

where $\Psi_n(k,m,z)$ are the axial standing-wave functions given by

$$\Psi_n(k,m,z) = \frac{1}{2} \left[\frac{2R_m - R_k}{S_n^2 - (R_m - R_k)^2} \cos (R_m - R_k) \beta_p z + \frac{2R_m + R_k}{S_n^2 - (R_m + R_k)^2} \cos (R_m + R_k) \beta_p z \right] . \tag{42}$$

The solution (41) will, of course, not satisfy our initial conditions at the plane $z = 0$. The conditions are $y = 0$ and

$$v_{1y} = \dot{y} + \dot{z}\,\frac{\partial x}{\partial z} = \frac{1}{4}\,v_o\left(\frac{v_1^o}{v_o}\right)^2 (\cos 2\omega t - 1) \quad ,$$

if we assume that the beam is velocity modulated in an infinitely short gridded gap at $z = 0$. The voltage across the gap is $V_1^o \sin \omega t$. If V_o is the d-c accelerating voltage, then $v_1^o/v_o = V_1^o/2V_o$. One can satisfy the initial conditions by writing

$$y = y_{nonhom} + y_{hom} \quad , \tag{43}$$

where y_{hom}, a solution to the homogeneous part of Equation (38), is chosen so that the initial conditions are satisfied. One finds that y_{hom}, has to be of the form of Equation (41) with $\Phi_n(k, m, z)$ replaced by

$$\phi_n(k, m, z) = -\frac{1}{2}\left[\frac{2R_m - R_k}{S_n^2 - (R_m + R_k)^2}\cos(S_n\beta_p z) + \frac{2R_m + R_k}{S_n^2 - (R_m + R_k)^2}\cos(S_n\beta_p z)\right] . \tag{44}$$

Thus the proper solution y becomes

$$y = \frac{1}{2}\left(\frac{v_1^o}{v_o}\right)^2 \frac{\beta_e}{\beta_e^2}\sin 2(\omega t - \beta_e z)\sum_{n=1}^{\infty}\left[\sum_{k,m}^{\infty} \epsilon_n(k, m)\,\epsilon_k\epsilon_m\,\frac{1}{R_m}\,\xi_n(k, m, z)\right] J_o(D_n r) \tag{45}$$

where

$$\xi_n(k, m, z) = \Psi_n(k, m, z) + \phi_n(k, m, z) = \frac{1}{2}\left\{\frac{2R_m - R_k}{S_n^2 - (R_m - R_k)^2}\left[\cos(R_m - R_k)\beta_p z\right.\right.$$

$$\left.\left. - \cos S_n\beta_p z\right] + \frac{2R_m + R_k}{S_n^2 - (R_m + R_k)^2}\left[\cos(R_m + R_k)\beta_p z - \cos S_n\beta_p z\right]\right\} . \tag{46}$$

Actually the solution (45) corresponds to the initial condition $v_{1y} = 0$ and not to

$$v_{1y} = \frac{1}{4} v_o \left(\frac{v_1^o}{v_o}\right)^2 (\cos 2\omega t - 1) \quad .$$

This is due to the fact that the latter condition would produce additional terms, which are of the order β_p/β_e times less than those present in Equation (45). Such terms have been neglected throughout our work on account of the assumption $\beta_p \ll \beta_e$. Thus the significance of the modulating gap nonlinearities is of the order β_p/β_e times less than the significance of the nonlinearities of the drifting beam.

The normalized second-order a-c current density becomes

$$\boxed{\frac{i_{1y}}{i_o} = \frac{\rho_o \frac{\partial y}{\partial t}}{\rho_o v_o} = \left(\frac{v_1^o \beta_e}{v_o \beta_p}\right)^2 \cos 2(\omega t - \beta_e z) \sum_{n=1}^{\infty}\left[\sum_{k,m} \epsilon_n(k, m)\, \epsilon_k \epsilon_m \frac{1}{R_m} \xi_n(k, m, z)\right] J_o(D_n r)} \; , \tag{47}$$

while the normalized total second-order a-c current in the beam is

$$\boxed{\frac{I_{1y}}{I_o} = \left(\frac{v_1^o \beta_e}{v_o \beta_p}\right)^2 \cos 2(\omega t - \beta_e z) \sum_{n=1}^{\infty} \left[\sum_{k,m} \epsilon_n(k,m) \epsilon_k \epsilon_m \frac{1}{R_m} \xi_n(k,m,z) \right] \frac{J_1(D_n b)}{\frac{D_n b}{2}}} \quad . \tag{48}$$

V. DISCUSSION

Equations (45), (47) and (48) represent the desired formal results. Expressions for other second-order wave quantities, such as v_{1y} can now be easily obtained by the use of Equation (45).

The results do not reveal their physical significance very easily, since they are expressed in the form of complex series. Further investigations must be made numerically on the basis of specific cases. This would include, for instance, a study of the sum $\sum_{n=1}^{\infty}$ in Equation (48) as a function of z for some typical beam and drift-tube geometries. We hope to return to this problem in a subsequent report.

However, some comments should be made concerning the axial standing-wave functions, $\xi_n(k, m, z)$, expressed in Equation (46). First we notice that $\xi_n(k, m, z)$ is an oscillatory but in general nonperiodic function of z for any fixed combination n, k, m, while the corresponding function of the linear theory, $\sin(R_k \beta_p z)$, is periodic. This means, for example, that even if we limit ourselves to a lowest "mode" study ($n = k = m = 1$), the second-harmonic standing-wave pattern will not repeat itself periodically along the beam. The periodic stand-wave pattern is a small-signal property which does not exist at large-signal levels when the beam is radially finite. The wave functions $\xi_n(k, m, z)$ can also be written in the form,

$$\xi_n(k, m, z) = (2R_m+R_k)\frac{1}{2}\beta_p z \; \frac{\sin\frac{1}{2}(R_m+R_k-S_n)\beta_p z}{\frac{1}{2}(R_m+R_k-S_n)\beta_p z} \cdot \frac{\sin\frac{1}{2}(R_m+R_k+S_n)\beta_p z}{(R_m+R_k+S_n)}$$

$$+ (2R_m-R_k) \frac{\sin\frac{1}{2}(R_m-R_k-S_n)\beta_p z}{R_m-R_k-S_n} \cdot \frac{\sin\frac{1}{2}(R_m-R_k+S_n)\beta_p z}{R_m-R_k+S_n} \quad . \tag{49}$$

Now, if $R_m+R_k-S_n \ll 1$, then $R_m+R_k-S_n \; \beta_p z \lessapprox \pi/2$ even for comparatively large $\beta_p z$. Under this condition,

$$\frac{\sin\frac{1}{2}(R_m+R_k-S_n)\beta_p z}{\frac{1}{2}(R_m+R_k-S_n)\beta_p z} \approx 1 \quad , \tag{50}$$

and the first term in Equation (49) will contain a factor proportional to z. The second term in Equation (49) will contain such a linearly growing factor when $\left|R_m-R_k-S_n\right| \ll 1$ or $R_m-R_k+S_n \ll 1$. However, the condition $\left|R_m+R_k-S_n\right| \ll 1$ is of particular interest, because it will be systematically fulfilled when $m = k = n \geqslant 2$ and $\beta_e b \lessapprox 3$ (see Olving[11] Figure 40). This is so because the plasma frequency reduction factors for the higher order modes are approximately proportional to the signal frequency when the beam circumference parameter $\beta_e b$ is in the practical range ($\lessapprox 3$). Thus, when $m = k = n \geqslant 2$, we will have $R_m = R_k \approx 1/2 \; S_n$, and the driving term $a(x)$, of Equation (33) will contain terms proportional to $\cos (R_m = R_k)\beta_p z = \cos S_n \beta_p z$. Such terms are solutions to the homogeneous part of Equation (33). Thus these particular driving terms are in resonance with the natural plasma oscillations of the beam, which, as we have seen, results in a standing-wave pattern growing linearly with distance. The conclusion is that the higher order terms, $m = k = n \gg 2$, will be of paramount importance when $\beta_p z$ becomes large.

If the beam is thin enough, say $\beta_e b \ll 1$, and so short that the growing higher-order "modes" can be ignored, then one should expect to get an adequate representation of the series appearing in Equations (45), (47) and (48) by including only the fundamental "mode" term $m = k = n = 1$. Furthermore, if $\beta_e b \ll 1$, one has $\epsilon_1 \simeq 1 \simeq \epsilon_1(1, 1)$ and $J_o(D_n r) \simeq 1$, which yield

$$y \simeq \frac{1}{2} \left(\frac{v_1^o}{v_o}\right)^2 \frac{\beta_e}{\beta_p^2} \frac{1}{R_1} \; \xi_1 \,(1, 1, z) \sin 2(\omega t - \beta_e z) \quad , \tag{51}$$

where

$$\xi_1\,(1, 1, z) = \frac{\frac{3}{2} R_1}{2R_1 + S_1} \; \beta_p z \; \frac{\sin\,(R_1 - \frac{1}{2} S_1)\,\beta_p z}{(R_1 - \frac{1}{2} S_1)\,\beta_p z} \cdot \sin(R_1 + \frac{1}{2} S_1)\,\beta_p z$$

$$+ \frac{R_1}{S_1^2} \; \sin^2 (\tfrac{1}{2} S_1 \beta_p z) \quad . \tag{52}$$

Since $S_1 \simeq 2R_1$, for thin beams one can write relation (51) in the form

$$y = \frac{1}{16} \left(\frac{v_1^o}{v_o}\right)^2 \frac{\beta_e}{(R_1 \beta_p)^2} \Big[1 - \cos\,(2R_1 \beta_p z) +$$

$$3 R_1 \beta_p z \sin (2R_1 \beta_p z) \Big] \cdot \sin 2(\omega t - \beta_e z) \quad . \tag{53}$$

Relations (51) and (53) are identical to the results obtained and discussed by Paschke.[2] This proves that Paschke's generalized procedure (see Appendix) for the use of the linear theory plasma-frequency reduction factors at nonlinear levels is perfectly justified provided the beam is thin enough, that is, $\beta_e b \ll 1$.

If the beam is so thin that the space-charge forces can be neglected, that is $R_1 \longrightarrow 0$, then relation (53) becomes

$$y \simeq \frac{1}{2}\left(\frac{v_1^o}{v_o}\right)^2 \beta_e z^2 \sin 2(\omega t - \beta_e z) \quad , \tag{54}$$

which, as expected, is identical to the corresponding result of Webster's[13] ballistic klystron theory.

Finally we want to show that Equation (45) is correct when the beam is very thick, that is, $\beta_e b \gg 1$ and $R_k = R_m = S_n = 1$. One immediately finds that in this case

$$\xi_n(k, m, z) = \sin^2 \beta_p z \quad , \tag{55}$$

and

$$\sum_{n=1}^{\infty}\left[\sum_{k,m} \epsilon_n(k, m)\, \epsilon_k \epsilon_m\right] J_o(D_n r) \equiv \left[\sum_{k=1}^{\infty} \epsilon_k\, J_o(T_k r)\right]^2 \quad . \tag{56}$$

The right-hand side of relation (56) is the square of the Fourier-Bessel expansion of the constant 1. Using relations (55) and (56) in Equation (45), one gets for the case of the infinitely thick beam,

$$y = \frac{1}{2}\left(\frac{v_1^o}{v_o}\right)^2 \frac{\beta_e}{\beta_p^2} \sin^2 \beta_p z \cdot \sin 2(\omega t - \beta_e z) \quad . \tag{57}$$

Equation (57) is, as expected, identical to the corresponding result obtained in connection with a study of nonlinear space-charge waves in radially infinite beams.[5] If we ignore the space-charge waves in relation (57), i.e., if we let $\beta_p \rightarrow 0$, we will again get the ballistic result in Equation (54).

VI. CONCLUSIONS

The nonlinear, nonrelativistic space-charge wave equation (12) for a well-confined radially finite homogeneous electron beam was deduced under the assumption of no electron overtaking, no velocity distribution, and no collisions. The first assumption is a serious one, since it does not allow one to deal reliably with questions related to saturation effects in very high-power tubes. In fact, overtaking is really the phenomenon that causes saturation.[5, 17, 18] The word "saturation" is used to describe the situation where a small increase of the input drive yields no increase of the output power.

Unfortunately the analytical description of space-charge waves beyond overtaking requires the solution of a complicated system of coupled, simultaneous, nonlinear, differential equations. Apart from Webster's[13] ballistic theory, the only analytical approach which allows overtaking seems to be an unpublished study by Roe.[18] Roe's work, however, is limited to the case of radially infinite beams. Some interesting computer results in the region beyond overtaking for the so-called disk-model beam have been obtained by Webber.[17, 19] Thus one of the really important challenges of the present-day microwave electronics is the development of an analytical, nonlinear, finite-beam, space-charge wave theory in the range past crossover.

It is worth while recalling that our nonlinear wave equation, although limited by the requirement of nonovertaking, is perfectly general in the sense that it governs the low-level nonlinear behavior of all kinds of

O-type space-charge-wave devices, e.g., traveling-wave tubes, klystrons, etc. The difference between the various tubes lies entirely in the boundary conditions. The proper solutions to the wave equation would describe such low-level nonlinear phenomena as the excitation of higher-order harmonic and nonlinear fundamental-frequency waves, output phase shift as a function of drive, d-c state of the beam as a function of drive, etc. In principle one would be able to predict the dynamic state of the beam for any modulating signal in the noncrossover region completely.

The purpose of the present report was to show that the nonlinear space-charge wave equation can be solved with the ''conducting tunnel'' boundary conditions (klystron). The attack was based on the method of successive approximations and the use of proper Fourier-Bessel expansions. The solution was worked out to the second order describing the second-order harmonic (2ω) under the assumption $\beta_p \ll \beta_e$. It was found that the lowest ''mode'' of the second harmonic (and all lowest ''modes'' of the higher-order harmonics) as a result contains, with distance, growing waves when the beam is thin ($\beta_e b \not> 1$). Higher-order nonlinear ''modes'' become important in thick beams (say $1 < \beta_e b < 3$) because the lowest ''modes'' associated with the various frequencies do not grow in the thick beam while certain higher-order ''modes'' still do. Contrary to the linear modes of the radially finite beam, and the linear and nonlinear modes of the radially infinite beam, the nonlinear finite beam ''modes'' do not repeat their standing-wave patterns periodically along the beam. However, the plasma-frequency reduction factors, well-known from the linear space-charge-wave theories, play a significant role in the nonlinear analysis.

It is finally shown in the Appendix that Paschke's recent theory[2, 7] is the correct nonlinear equivalent to the linear single-mode space-charge-wave theory of Ramo.[20]

APPENDIX

The purpose of this appendix is to show that the nonlinear wave equation used by Paschke[2,7] in its region of validity is in agreement with the exact Equation (12).

Assume that the beam is so thin $(\beta_e b << 1)$ that the oscillatory properties of the electrons at the center and the edge are essentially the same. This means that radial wave functions can be ignored. Thus z_1 is a function of time and axial distance z only. Since overtaking is not permitted, z_1 is a one-valued function. Furthermore, z_1 is periodic in time provided the initial disturbance is periodic. Thus z_1 can be expanded in a Fourier series,

$$z_1(z, t) = \sum_{n=0}^{\infty} Z_{n\omega}(z, t) \quad , \qquad \text{(A. 1)}$$

where $Z_{n\omega}$ can be written in complex notations as

$$Z_{n\omega}(z, t) = a_n(z)\, e^{jn(\omega t - \beta_e z)} \quad . \qquad \text{(A. 2)}$$

The problem is to determine the slowly varying functions $a_n(z)$ with reasonable accuracy. For that reason one has to find an approximate wave equation for z_1, independent of r.

The exact wave equation is Equation (12) which can be written,

$$(P + Q)\ \frac{d^2 z_1}{dt^2} + Q\omega_p^2\, z_1 = 0 \quad . \tag{A. 3}$$

If we use Equation (A. 1) in (A. 3) and ignore the radial variations (i. e., P = 0) we will get

$$\sum_{n=0}^{\infty} \left(\frac{d^2}{dt^2} + \omega_p^2\right) z_{n\omega}(z, t) = 0 \quad . \tag{A. 4}$$

Equation (A. 4) obviously describes the radially infinite beam and not the thin beam we are interested in. In the thin beam the effect of the fringe field is extremely important and has to be taken into account. Paschke assumes that this can be done by introducing plasma-frequency reduction factors into Equation (A. 3) by writing,

$$\boxed{\sum_{n=0}^{\infty} \left(\frac{d^2}{dt^2} + R_{n\omega}^2\ \omega_p^2\right) Z_{n\omega}(z, t) = 0 \ ;} \tag{A. 5}$$

$R_{n\omega}$ is supposed to be identical to the linear theory plasma-frequency reduction factor evaluated for the lowest mode at the frequency $n\omega$. Thus $R_{o\omega} = 0$, while $R_{1\omega} = R_1$ and $R_{2\omega} = S_1$ in our earlier notations. Equation (A. 5) is a generalization of Paschke's Equations (4a, b, c)[7] which are limited to the third-order accuracy. The construction of Equation (A. 5) is supported by the fact that it is definitively correct when the equation is linearized. However, one may question whether it is strictly valid in the nonlinear case. In what follows we will show that Equation (A. 5) can indeed be deduced from the exact Equation (A. 3).

Let us write

$$\frac{d^2 z_1}{dt^2} = \sum_{n=0}^{\infty} X_{n\omega}(z, t) \quad ; \tag{A. 6}$$

i. e., we expand the acceleration in Fourier series; $X_{n\omega}$ is of the form

$$X_{n\omega}(z, t) = b_n(z) e^{jn(\omega t - \beta_e z)} \quad . \tag{A. 7}$$

Since $\beta_p \ll \beta_e$, we can expect the amplitude functions $b_n(z)$ to vary very slowly compared to $\exp(-jn\,\beta_e z)$.

We will now insert Equations (A. 1) and (A. 6) into Equation (A. 3). If $\beta_e b \ll 1$, then both $Z_{n\omega}$ and $X_{n\omega}$ contain a factor $J_o(T_{n\omega} r)$, the radial wave function of the lowest "mode" at the frequency $n\omega$ (see Chapters III and IV). The quantity $T_{n\omega}$ is determined by the radial impedance matching relation [see Equations (27) and (39)]:

$$T_{n\omega} b \, \frac{J_1(T_{n\omega} b)}{J_o(T_{n\omega} b)} = F(z, b, n\beta_e) \quad ; \tag{A. 8}$$

$T_{n\omega}$ is simply the linear theory lowest-mode radial propagation constant evaluated for the frequency $n\omega$. Thus $T_{o\omega} = 0$ while $T_{1\omega} = T_1$ and $T_{2\omega} = D_1$, in our earlier notations. When $\beta_e b \ll 1$ then $J_o(T_{n\omega} r) \simeq 1$, while $P J_o(T_{n\omega} r) \simeq - T_{n\omega}^2$.

We make use of the result $P \simeq - T_{n\omega}^2$ and Equations (A. 1) and (A. 6) in (A. 3) to obtain

$$\sum_{n=0}^{\infty} \left[\left(- T_{n\omega}^2 - n^2 \beta_e^2 \right) X_{n\omega} + \left(- n^2 \beta_e^2 \omega_p^2 \right) Z_{n\omega} \right] = 0 \quad . \tag{A. 9}$$

Collecting terms of equal frequency in Equation (A. 9) and equating to zero, we get

$$X_{n\omega} + \frac{\omega_p^2}{1 + \left(\frac{T_{n\omega} b}{n\beta_e b} \right)^2} Z_{n\omega} = 0 \quad , \tag{A. 10}$$

where one recognizes

$$R_{n\omega}^2 = \frac{1}{1 + \left(\frac{T_{n\omega} b}{n\beta_e b} \right)^2} \quad . \tag{A. 11}$$

Thus one has

$$X_{n\omega} + R_{n\omega}^2 \, \omega_p^2 \, Z_{n\omega} = 0 \tag{A. 12}$$

or

$$\sum_{n=0}^{\infty} \left(X_{n\omega} + R_{n\omega}^2 \omega_p^2 \, Z_{n\omega} \right) = 0 \quad , \tag{A. 13}$$

which is the same as Equation (A. 5). Third-order solutions to Equation (A. 5) have been obtained by Paschke,[7] Romaine,[14] Blair,[15] and Engler.[16]

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HARMONIC ANALYSIS OF ELECTRON BEAMS IN KLYSTRONS

W. E. Blair

School of Electrical Engineering
CORNELL UNIVERSITY
Ithaca, New York

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W. E. Blair

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OF LINEAR-BEAM MICROWAVE TUBES

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ABSTRACT

This analysis describes the modulation and the electron bunching processes of klystrons. The final velocity of the interaction region or buncher gap is derived including the first and second order effects of transit time. The result is a final velocity which is an infinite series of time harmonics, and the coefficient of each harmonic is a product of Bessel functions whose arguments are determined by the physical and operational parameters of the klystron. This result of nonlinearities shows that the gap-coupling coefficient is no longer a simple constant but is a function of the amplitude of modulation and buncher gap width.

The nonlinear bunching process occurring in the drift space is described by a nonlinear differential equation which is solved by the method of successive approximations. The solution reveals current and velocity harmonics which are nonperiodic in space. The higher harmonics are dependent in part on the lower harmonics, and all harmonics above the fundamental have irrationally decreasing amplitudes and periods. The envelope of the higher harmonics grows in amplitude initially with increase in distance along the drift space. This instability can be interpreted as the parametric transfer of energy from the lower harmonics to higher harmonics. By using an interaction impedance, a correction to the usual computation of phase velocity of the fast and slow space-charge waves is obtained. This correction causes the velocity to have a minimum, never a null because the unequal amplitude of the waves can never cancel each other. The current at the fundamental and harmonic frequencies have the expected nulls, but are slightly displaced in space.

INTRODUCTION

Since Webster[1] first described the bunching process of an electron beam in a klystron, more and more accurate descriptions of this process have been sought. This has been especially true as higher frequencies and higher powers became more predominant. Because of these two factors, the transit time of the electrons passing through the interaction region has become a significant limitation to the accuracy of theories of klystron modulation which ignore second-order transit-time effects in the buncher gap. This analysis attempts to describe more accurately the velocity modulation process of electrons passing through the buncher cavity.

The transit time analysis does not include space-charge effects because there is very little bunching in the interaction region; however, any accurate drift space analysis must include space-charge effects, especially for large-signal operation. The bunching process of electron beams is described by a nonlinear equation that is extremely difficult to solve. Paschke[2] has used the method of successive approximations to solve this equation; consequently, one can describe the spatial variations of convection-current density and velocity harmonics for a fintie electron beam.

Although a one dimensional model of the electron beam is used, this solution reveals several interesting properties of space-charge waves. These properties have been verified by Mihran[8] and others. This analysis becomes very lengthy when used to calculate higher harmonics, but can always be used for any harmonic desired. The method is used in this paper to extend Paschke's work to a third-order solution including the third harmonic and connections to the fundamental in a finite beam. This general method can be extended to more complicated beam models, even a model allowing radial variations.

A correction to the phase velocity of the fast and slow space-charge waves can be obtained deriving an interaction impedance used by McIsaac and Wang[6]. This reveals shifts in current-amplitude variations with space and time, resulting in spatially displaced current nulls and additional phase shifts in the current components.

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I. TRANSIT-TIME CORRECTION TO THE VELOCITY IN BUNCHER REGION

Symbols Used in This Section:

a	Acceleration
e	Electronic charge (magnitude only)
E_{cz}	Circuit field
m	Electronic mass
t	Any arbitrary time in the interaction region, such that $t_o \leq t \leq t_f$
t_o	Time in the applied voltage cycle at which the electron enters the buncher gap
t_f	Time in the applied voltage cycle at which the electron exits from the buncher gap
u_o	D-C velocity of electrons
u_f	Total final velocity of electrons
$\tilde{u}_f$	A-C final velocity of electrons
V	Applied a-c voltage (accelerating)
V_o	D-C beam voltage corresponding to velocity u_o
V_{eff}	Effective voltage across buncher gap
z	Longitudinal direction
β_e	Electronic phase constant = ω/u_o
ϵ_o	Capacitivity of free space
η	Ratio e/m
ω	Angular frequency of applied voltage (buncher cavity resonant frequency)

The purpose of this section is to describe the velocity in the buncher gap or interaction region of a klystron, including the effect of transit time. It is necessary that a complete theory involving the transit time consider both a finite buncher gap width and a changing electric circuit field while the electron passes through the interaction region.

The only assumptions made for the model of the klystron interaction region are the following:

1. The electric field remains constant over the cross section of the klystron beam.

2. No space charge will be considered within the interaction region.

3. The klystron beam will be considered to be uniform and to move in confined flow.

The model for the buncher gap used throughout the remainder of this paper is as follows:

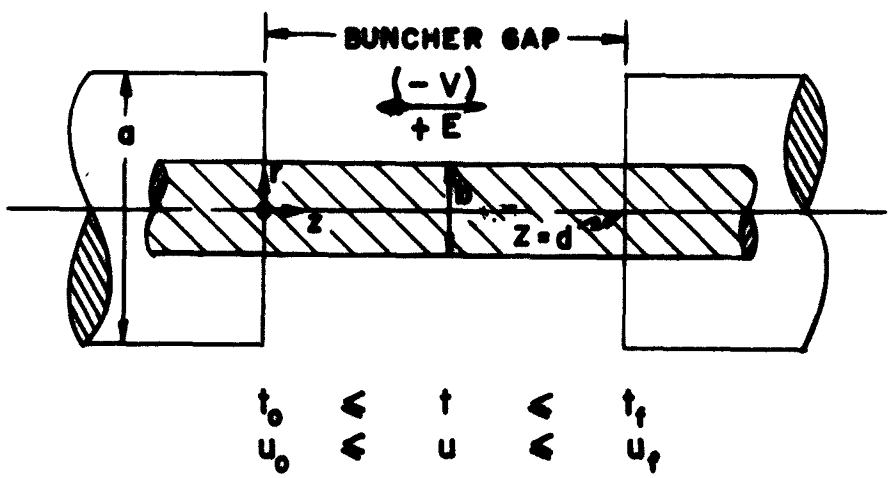

Figure 1. Buncher Gap or Interaction Region.

From the force equation, the acceleration of an electron through the interaction region can be written as

$$a = -\eta E_{cz} = \eta \frac{V}{d} \cos \omega t \quad . \tag{1}$$

Integrating the acceleration gives the velocity at any arbitrary time, t, in the buncher gap as

$$u = \frac{\eta}{\omega} \frac{V}{d} \left(\sin \omega t - \sin \omega t_o \right) + u_o \quad . \tag{2}$$

In order to know the exact velocity of an electron at the end of the buncher gap, it is necessary to express the velocity as a function of the longitudinal distance, z. Subsequently, the velocity can be evaluated at the end of the gap, $z = d$. One means of accomplishing this is to integrate the velocity to give the distance as a function of time, solve for time, and substitute for the velocity of Equation (2). Integrating the velocity gives the distance as

$$z = -\frac{\eta}{\omega^2} \frac{V}{d} \left(\cos \omega t - \cos \omega t_o \right) + \left(-\frac{\eta}{\omega} \frac{V}{d} \sin \omega t_o + u_o \right) (t - t_o) \quad . \tag{3}$$

However, it is impossible to solve directly for time as a function of distance because Equation (3) is a transcendental equation. One very accurate approximation is to solve for t as a function of z and $\cos \omega t$, and substitute for ωt as a function of distance:

$$\omega t = \omega t_o + \beta_e z \quad . \tag{4}$$

Consequently, the time can be written as

$$t = \frac{z + \frac{\eta}{\omega^2}\frac{V}{d}\left[\cos(\omega t_o + \beta_e z) - \cos\omega t_o\right]}{-\frac{\eta}{\omega}\frac{V}{d}\sin\omega t_o + u_o} + t_o \quad . \tag{5}$$

Evaluating Equation (5) at $z = d$, gives the final time, t_f. Substituting Equation (5) into Equation (2) gives the final normalised velocity u_f/u_o as

$$\frac{u_f}{u_o} = 1 + \frac{\alpha}{2}\frac{1}{\beta_e d}\left\{\sin\left[\frac{\omega d + \frac{\alpha}{2}\frac{u_o}{\beta_e d}\left[\cos(\omega t_o + \beta_e d) - \cos\omega t_o\right]}{u_o\left(1 - \frac{\alpha}{2}\frac{1}{\beta_e d}\sin\omega t_o\right)} + \omega t_o\right] - \sin\omega t_o\right\}, \tag{6}$$

where

$$\frac{1}{u_o}\frac{\eta}{\omega}\frac{V}{d} = \frac{\alpha}{2}\frac{1}{\beta_e d} \quad , \tag{7}$$

and

$$\alpha = \frac{V}{V_o} \quad .$$

Equation (6) is an expression for the final velocity of the buncher gap as a function of the entrance time and gap width. Consequently, onc the initial phase of the drive voltage, V, is fixed, the final velocity of all electrons entering at that instant is determined. However, for later use in this section, it is convenient to write the final velocity as a function of the exit time of the electrons. Using half-angle trigonometric identities and letting $\omega t_o = \omega t_f - \beta_e d$, the variational component of the final velocity can be written as

$$\frac{u_f}{u_o} = \frac{\alpha}{2} \; \frac{\sin\left[\dfrac{\frac{1}{2}\beta_e d + \frac{\alpha\mu}{4}\sin\left(\omega t - \frac{\beta_e d}{2}\right)}{1 + \frac{\alpha}{2}\frac{1}{\beta_e d}\sin(\omega t - \beta_e d)}\right]}{\dfrac{\beta_e d}{2}}$$

$$\cos\left[\omega t - \beta_e d + \frac{\frac{1}{2}\beta_e d + \frac{\alpha\mu}{4}\sin\left(\omega t - \frac{\beta_e d}{2}\right)}{1 + \frac{\alpha}{2}\frac{1}{\beta_e d}\sin(\omega t - \beta_e d)}\right] , \qquad (8)$$

where

$$\mu = \frac{\sin\dfrac{\beta_e d}{2}}{\dfrac{\beta_e d}{2}} .$$

From Equation (8) it is possible to expand the final velocity giving an infinite series of time harmonics. This, however, is not as accurate as the following approach where Equation (8) is used to calculate a transit time; that is, the final velocity can be written as a function of the entrance time and transit time through the buncher gap.

The final velocity, from Equation (2), can be written as a function of the transit time, τ, giving

$$u_f = \frac{\eta}{\omega}\frac{V}{d}\left[\sin\omega(t_o + \tau) - \sin\omega t_o\right] + u_o . \qquad (9)$$

The transit time can be defined exactly using the average velocity, $\bar{u}$, but the average velocity can be approximated very closely to $\frac{1}{2}(u_o + u_f)$.

Thus the transit time can be written as

$$\tau = \frac{d}{\bar{u}} = \frac{d}{\frac{1}{2}(u_o + u_f)} = \frac{d}{u_o}\left(\frac{1}{1+\delta}\right) \quad , \tag{10}$$

where

$$\delta = \frac{1}{2}\,\frac{u_f}{u_o} \quad .$$

This δ represents the correction to the average beam velocity, u_o, resulting from the modulation voltage. Substituting Equation (10) into Equation (9) and writing in the form of Equation (8), yields

$$\frac{u_f}{u_o} = 1 + \frac{a}{2}\,\frac{\sin\left[\frac{\beta_e d}{2}\left(\frac{1}{1+\delta}\right)\right]}{\frac{\beta_e d}{2}}\cos\left[\omega t - \beta_e d + \frac{\beta_e d}{2}\left(\frac{1}{1+\delta}\right)\right]. \tag{11}$$

Comparing Equations (8) and (11), one can easily see that δ is approximately

$$\delta = \frac{a\mu}{4}\cos\left(\omega t - \frac{\beta_e d}{2}\right), \quad . \tag{12}$$

It is worth noting that both the correction to the final time in Equation (8) and the transit time will give the same order correction to the final velocity. This accounts for the entrance phase of the field being different, depending upon the time at which the electrons enter. However, it is shown in Appendix A that Equation (11) can be adjusted to give a second-order (or higher) correction to the final velocity, by replacing δ with $\frac{\delta}{1+\delta}$. Using the binominal expansion gives the final velocity

$$\frac{u_f}{u_o} = 1 + \frac{a}{2} \frac{\sin\left[\frac{\beta_e d}{4} (1 - \delta + 2\delta^2)\right]}{\frac{\beta_e d}{2}} \cos\left[\omega t - \beta_e d + \frac{\beta_e d}{2} (1 - \delta + 2\delta^2)\right] . \tag{13}$$

This second-order correction represents physically the changing of the modulating voltage while the electrons are passing through the buncher gap.

Substituting Equation (12) into Equation (13) , one can write the final velocity as an infinite series of time harmonics whose coefficients are Bessel functions. The series is of the general form

$$\frac{u_f}{u_o} = a_o + \sum_n a_n \cos n\omega t + \sum_n b_n \sin n\,\omega t \quad . \tag{14}$$

Within the limitations of the original assumptions, this series based on Equation (13) will be exact through the third harmonic. The complete series is given in Appendix B, but the main terms are

$$\frac{u_f}{u_o} = 1 + \frac{a}{2} \frac{1}{\frac{\beta_e d}{2}} \left\{ - \cos\frac{\beta_e d}{2} J_o\left(\frac{\beta_e d}{2} \frac{a\mu}{4}\right) J_1\left(\frac{\beta_e d}{2} \frac{a\mu}{4}\right) \right.$$

$$+ \left[\sin\frac{\beta_e d}{2} J_o^2\left(\frac{\beta_e d}{2} \frac{a\mu}{4}\right) + 3\cos\frac{\beta_e d}{2} J_o^2\left(\frac{\beta_e d}{2} \frac{a\mu}{4}\right) J_1\left(\frac{\beta_e d}{2}\left[\frac{a\mu}{4}\right]^2\right)\right]$$

$$\cos\left(\omega t - \frac{\beta_e d}{2}\right)$$

$$- \left[\cos\frac{\beta_e d}{2} J_o\left(\frac{\beta_e d}{2} \frac{a\mu}{4}\right) J_1\left(\frac{\beta_e d}{2} \frac{a\mu}{4}\right)\right] \cos 2\left(\omega t - \frac{\beta_e d}{2}\right)$$

$$+\left[\cos\frac{\beta_e d}{2}\, J_o^{\,2}\left(\frac{\beta_e d}{2}\frac{a\mu}{4}\right) J_1\left(\frac{\beta_e d}{2}\left[\frac{a\mu}{4}\right]^2\right)\right] \cos 3\left(\omega t - \frac{\beta_e d}{2}\right)\Bigg\} \quad . \tag{15}$$

Equation (14) can be reduced to Webster's[1] ballistic theory for the velocity by considering an infinitesimally thin gap; in this case, the final velocity reduces to

$$\frac{u_f}{u_o} = 1 + \frac{a}{2}\cos\omega t - \frac{a^2}{16}(1 + \cos 2\omega t) + \frac{a^3}{64}(3\cos\omega t + \cos 3\omega t) \quad . \tag{16}$$

To best illustrate the effect that the correction to the transit time has on the final velocity, an example is chosen for maximum driving conditions for a klystron. Figure 2 shows the velocity as a function of $\omega t - \frac{\beta_e d}{2} = \phi$ with $a = 1$ and $\beta_e d = 2$. Since the physical and operational parameters for the klystron are specified, the series expansion using Equation (B2) becomes one with constant coefficients. The first three harmonics and their total are plotted.

The shape of the final velocity curve is basically a cosine wave correction to the d-c velocity as shown by Equation (13). The decrease in the d-c (average) level is caused by time variations in the gap-coupling coefficient. The skewing of the sine wave (shift in maximum and minimum) is caused by the time variations in the phase of the argument of the cosine wave. One significant result of this analysis is that the gap-coupling coefficient has become somewhat meaningless because it is no longer a constant, but a variable coefficient. In fact, it can be represented by an infinite series of time harmonics, whose coefficients are Bessel functions of $\beta_e d$ and a, in the same general form of Equation (14).

To illustrate the skewing, it is possible to examine the rate of change of the velocity. From Equation (13) the maximum deceleration is found to be

$$\frac{a\mu}{2}\left(1+\frac{\beta_e d}{2}\,\frac{a\mu}{4}\right) \quad , \tag{17}$$

while the maximum acceleration is

$$\frac{a\mu}{2}\left(1-\frac{\beta_e d}{2}\,\frac{a\mu}{2}\right) \quad . \tag{18}$$

If the gap is reduced to zero, the acceleration equals the deceleration, but if a finite gap exists, the deceleration is greater than the acceleration. This is clearly shown in Figure 2.

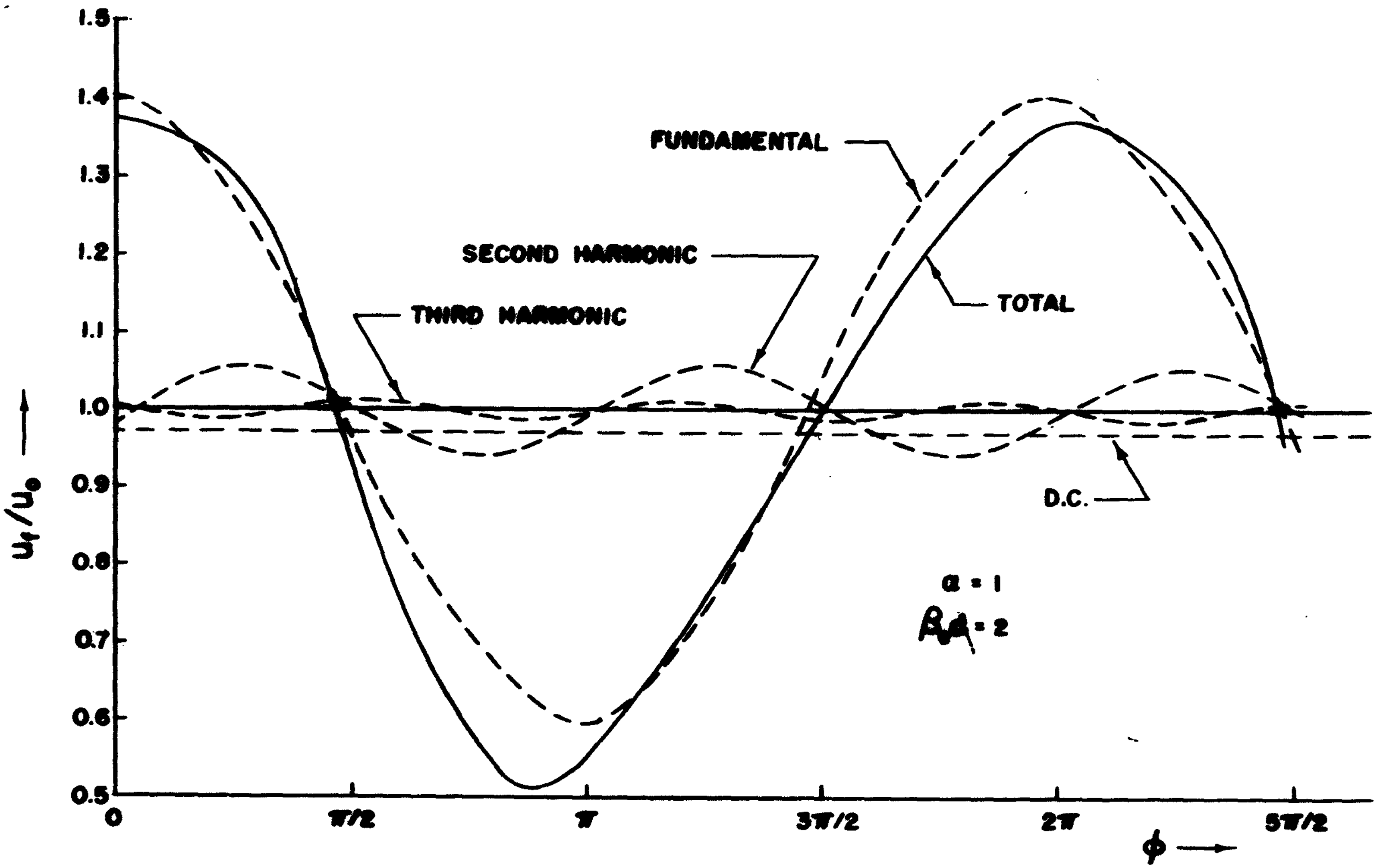

Figure 2. Normalized Final Velocity versus $(\omega t - \frac{1}{2}\beta_e d) = \phi$.

II. SPACE-CHARGE-WAVE HARMONIC ANALYSIS IN DRIFT REGION

Symbols Used in This Section

t	Transverse (subscript)
z	Longitudinal (subscript)
o	Direct-current (subscript)
$\underline{P}$	Eulerian polarization of electron beam
$\underline{E}$	Total electric field
$\underline{J}_c$	Convection current density
J_t	Displacement (transverse) current density
ρ	Volumetric charge density
θ	Relative phase of an electron beam perturbation and reference microwave circuit perturbation
θ_o	Initial relative phase of an electron beam perturbation and reference microwave circuit perturbation
ω_p	Plasma frequency
ω_q	Reduced plasma frequency
β_p	Plasma phase constant
β_q	Reduced plasma phase constant
R	Space-charge reduction factor

$$\xi_{mn} = \frac{\beta_m}{\beta_n}$$

One method of describing the nonlinear behavior of electron beams in klystrons is to consider the beam as a medium or fluid in which disturbances or perturbations to the beam propagate. In klystrons, velocity modulation of the beam produces density modulation, causing space-charge waves to propagate in the beam. The equation describing the nonlinear space-charge wave propagation can be solved by the method of successive approximations developed by Paschke.[2]

The assumptions made for this analysis are the following:

1. The one-dimensional confined flow (infinitely strong magnetic field) model of electron beam will be used.

2. The electron and phase velocities are small compared to the speed of light.

3. The longitudinal component of the r-f electric field is constant over the transverse area of the beam; that is, the transverse electric field varies linearly in the radial direction.

4. The effects of potential depression across the beam caused by space charge and variations in electron velocities caused by thermal noise are negligible.

Figure 3 shows the model of a finite, one-dimensional cylindrical beam in a cylindrical drift tube. The transverse displacement current density per meter J_t represents the loss of current because of fringing in a finite beam. This also accounts for the decrease in the longitudinal field E_z, and the longitudinal polarization P_z, the dependent variable, is given by

$$P_z = \epsilon_o E_z + \frac{\partial^2 E_t}{\partial z \, \partial t} \quad . \tag{19}$$

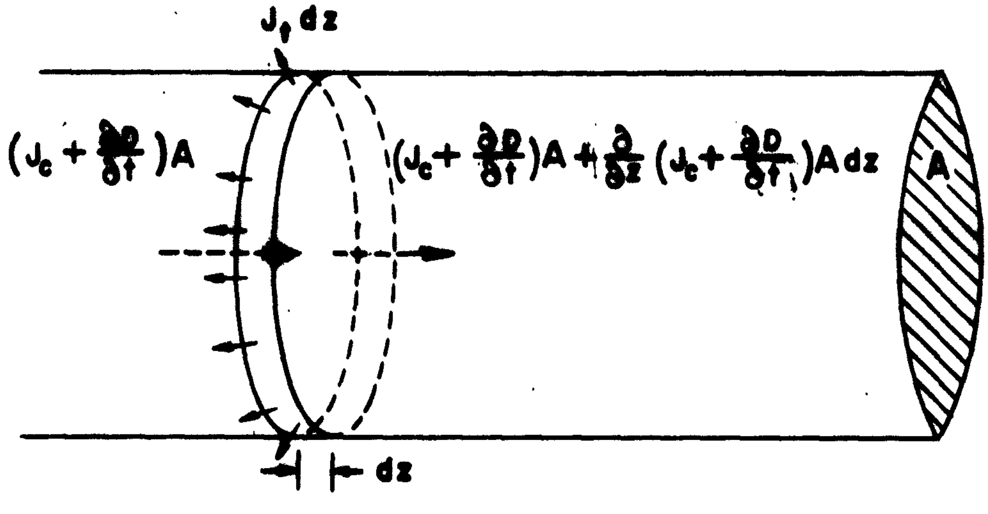

Figure 3. Current Flow Through Electron-Beam Model.

Thus in the infinite beam case, the polarization reduces to the longitudinal displacement D_z, since there is no fringing. Using the fundamental continuity equation

$$\nabla \cdot J = \frac{\partial J}{\partial z} = -\frac{\partial \rho}{\partial t} \quad , \tag{20}$$

one can define the polarization such that

$$-\frac{\partial P_z}{\partial t} = J \quad , \tag{21a}$$

$$\frac{\partial P_z}{\partial z} = \rho \quad . \tag{21b}$$

The continuity equation then can be written as

$$\frac{\partial J}{\partial z} + \epsilon_o \frac{\partial^2 E_z}{\partial z \, \partial t} + \frac{\partial^2 E_t}{\partial z \, \partial t} = 0 \quad , \tag{22}$$

where the term $\frac{\partial^2 E_t}{\partial z \, \partial t}$ represents the transverse coupling or fringe field. Two additional equations are needed; one is the convection current density, which is

$$J_o + J = (\rho_o + \rho)(u_o + \tilde{u}) \tag{23a}$$

$$J_o = \rho_o u_o \quad ; \tag{23b}$$

the other is the force equation, which is

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + (u_o + \tilde{u}) \frac{\partial u}{\partial z} = - \eta E_z \quad . \tag{24}$$

Combining Equations (19), (22), (23), and (24), one derives the following perturbation nonlinear differential equation:[2, 4]

$$\left(\frac{\partial^2 P_z}{\partial t^2} + u_o \frac{\partial^2 P_z}{\partial z \, \partial t}\right)\left(\rho_o + \frac{\partial P_z}{\partial z}\right)^2 - \frac{\partial^2 P_z}{\partial z \, \partial t} \left(\frac{\partial P_z}{\partial t} + u_o \frac{\partial P_z}{\partial z}\right) \left(\rho_o + \frac{\partial P_z}{\partial z}\right)$$

$$+ \left(J_o - \frac{\partial P_z}{\partial t}\right)\left(\frac{\partial^2 P_z}{\partial z \, \partial t} + u_o \frac{\partial^2 P_z}{\partial z^2}\right) \left(\rho_o + \frac{\partial P_z}{\partial z}\right)$$

$$- \left(J_o - \frac{\partial P_z}{\partial t}\right) \left(\frac{\partial P_z}{\partial t} + u_o \frac{\partial P_z}{\partial z}\right) \frac{\partial^2 P_z}{\partial z^2} + \frac{\eta}{\epsilon_o} \left(P_z - \frac{\partial^2 J_t}{\partial z \, \partial t}\right) = 0 \quad . \tag{25}$$

The dependent variable is the polarization and is a function of space and time and will have its value given as an initial condition. The second dependent variable with which initial conditions are satisfied is the velocity

$$\mathcal{V} = \frac{\dfrac{\partial P_z}{\partial t} + u_o \dfrac{\partial P_z}{\partial z}}{\rho_o + \dfrac{\partial P_z}{\partial z}} \quad . \tag{26}$$

Equation (25) can be rearranged in a more meaningful form, yielding

$$\frac{\partial^2 P_z}{\partial z^2} + \frac{2}{u_o}\frac{\partial^2 P_z}{\partial z\,\partial t} + \frac{1}{u_o^2}\frac{\partial^2 P_z}{\partial t^2} + \beta_q^2 P_z$$

$$= \frac{2}{J_o u_o}\left(\frac{\partial P_z}{\partial t}\frac{\partial^2 P_z}{\partial z\,\partial t} - \frac{\partial P_z}{\partial z}\frac{\partial^2 P_z}{\partial t^2}\right) + \frac{2}{J_o}\left(\frac{\partial P_z}{\partial t}\frac{\partial^2 P_z}{\partial t\, z^2} - \frac{\partial P_z}{\partial z}\frac{\partial^2 P_z}{\partial z\,\partial t}\right)$$

$$- 3\frac{\omega_p^2}{J_o u_o} R^2 P_z \frac{\partial P_z}{\partial z} - \frac{1}{J_o^2}\left[\frac{\partial^2 P_z}{\partial z^2}\left(\frac{\partial P_z}{\partial t}\right)^2 - 2\frac{\partial^2 P_z}{\partial z\,\partial t}\frac{\partial P_z}{\partial z}\frac{\partial P_z}{\partial t}\right.$$

$$\left. + \frac{\partial^2 P_z}{\partial t^2}\left(\frac{\partial P_z}{\partial z}\right)^2\right] - 3\frac{\omega_p^2}{J_o^2} R^2 P_z \left(\frac{\partial P_z}{\partial z}\right)^2 \quad . \tag{27}$$

where $\omega_p^2 = \eta \dfrac{\rho_o}{\epsilon_o}$, and $R = \dfrac{\omega_q}{\omega_p} = \dfrac{\beta_q}{\beta_p}$. The fringing of a finite beam is accounted for in Equation (25) by reducing the electric field as shown in Equation (19), but fringing is also accounted for by use of the space-charge reduction factor. Consequently, in Equation (27), the following relation is used:

$$P_z - \frac{\partial^2 E_t}{\partial z\,\partial t} = R^2 P_z \quad . \tag{28}$$

When the method of successive approximations is used, the solution is assumed to be

$$P_z = P_{z1} + P_{z2} + P_{z3} + \cdots \quad , \tag{29a}$$

$$u_z = u_{z1} + u_{z2} + u_{z3} + \cdots \quad , \tag{29b}$$

where P_{zn} and u_{zn} are the nth harmonic solution of the polarization and the velocity respectively. The method is essentially one in which the solution for P_{z1} is obtained ignoring higher harmonic solutions; then P_{z2} is obtained using P_{z1} and ignoring higher harmonic solutions; etc. In this case, Equation (27) is solved through the third harmonic. It is convenient to write Equation (27) as a sum of harmonic components as

$$f_1 + f_2 + f_3 \ . \ . \ . \ = 0 \quad , \tag{30}$$

such that each $f_n = 0$. This solution of Equation (30) is only one of many, but because it satisfies the boundary conditions, it is the complete solution. Substituting the solution of Equation (29) into Equation (27) and separating by harmonics, one obtains

for $f_1 = 0$:

$$\frac{\partial^2 P_1}{\partial z^2} + \frac{2}{u_o} \frac{2P_{z1}}{\partial z\, \partial t} + \frac{1}{u_o^2} \frac{\partial^2 P_{z1}}{\partial t^2} + \beta_{q1}^{\ 2} P_{z1} = 0 \quad ; \tag{31a}$$

for $f_2 = 0$:

$$\frac{\partial^2 P_{z2}}{\partial z^2} + \frac{2}{u_o} \frac{\partial P_{z2}}{\partial z\, \partial t} + \frac{1}{u_o^2} \frac{\partial P_{z2}}{\partial t^2} + \beta_{q2}^{\ 2} P_{z2} = \frac{2}{J_o u_o} \left(\frac{\partial P_{z1}}{\partial t} \frac{\partial^2 P_{z1}}{\partial z\, \partial t} - \frac{\partial P_{z1}}{\partial z} \frac{\partial P_{z1}}{\partial t^2} \right)$$

$$+ \frac{2}{J_o} \left(\frac{\partial P_{z1}}{\partial t} \frac{\partial^2 P_{z1}}{\partial z^2} - \frac{\partial^2 P_{z1}}{\partial z} \frac{\partial P_{z1}}{\partial z\, \partial t} \right) - 3 \frac{\omega_p^2}{J_o u_o} R_1^{\ 2} P_{z1} \frac{\partial P_{z1}}{\partial z} \quad ; \tag{31b}$$

for $f_3 = 0$:

$$
\begin{aligned}
&\frac{\partial^2 P_{z3}}{\partial z^2} + \frac{2}{u_o} \frac{\partial P_{z3}}{\partial z\, \partial t} + \frac{1}{u_o^2} \frac{\partial P_{z3}}{\partial t^2} + \beta_{q3}^{\ 2} P_{z3} \\
&= \frac{2}{J_o u_o} \left(\frac{\partial P_{z2}}{\partial t} \frac{\partial^2 P_{z1}}{\partial z\, \partial t} - \frac{\partial P_{z1}}{\partial t} \frac{\partial^2 P_{z2}}{\partial z\, \partial t} - \frac{\partial P_{z1}}{\partial z} \frac{\partial P_{z2}}{\partial t^2} - \frac{\partial^2 P_{z2}}{\partial z} \frac{\partial^2 P_{z1}}{\partial t^2} \right) \\
&+ \frac{2}{J_o} \left(\frac{\partial P_{z1}}{\partial t} \frac{\partial^2 P_{z2}}{\partial z^2} + \frac{\partial P_{z2}}{\partial t} \frac{\partial^2 P_{z1}}{\partial z^2} - \frac{\partial P_{z1}}{\partial z} \frac{\partial P_{z2}}{\partial z\, \partial t} - \frac{\partial P_{z2}}{\partial z} \frac{\partial P_{z1}}{\partial z\, \partial t} \right) \\
&- \frac{1}{J_o^{\ 2}} \left[\frac{\partial^2 P_{z1}}{\partial z^2} \left(\frac{\partial P_{z1}}{\partial t} \right)^2 - 2 \frac{\partial P_{z1}}{\partial t} \frac{\partial P_{z1}}{\partial z} \frac{\partial^2 P_{z1}}{\partial z\, \partial t} + \frac{\partial^2 P_{z1}}{\partial t^2} \left(\frac{\partial P_{z1}}{\partial z} \right)^2 \right] \\
&- 3 \frac{\omega_p^{\ 2}}{J_o u_o} \left(\frac{\partial P_{z2}}{\partial z} R_1^{\ 2} P_{z1} + R_2^{\ 2} P_{z2} \frac{\partial P_{z2}}{\partial z} \right) - 3 \frac{\omega_p^{\ 2}}{J_o u_o} R_1^{\ 2} P_{z1} \left(\frac{\partial P_{z1}}{\partial z} \right)^2 ;
\end{aligned}
\tag{31c}
$$

where β_{qn} and R_n are the space-charge-wave phase constant and space-charge reduction factor respectively for the nth space-charge harmonic.

The corresponding velocity equations for each harmonic component in Equation (29b) are

$$
u_{z1} = - \left[\frac{\partial}{\partial t} \left(\frac{P_{z1}}{\rho_o} \right) + u_o \frac{\partial}{\partial z} \left(\frac{P_{z1}}{\rho_o} \right) \right] \tag{32a}
$$

$$
u_{z2} = - \frac{\partial}{\partial t} \left[\left(\frac{P_{z2}}{\rho_o} \right) + u_o \frac{\partial}{\partial z} \left(\frac{P_{z2}}{\rho_o} \right) + u_{z1} \frac{\partial}{\partial z} \left(\frac{P_{z1}}{\rho_o} \right) \right] \tag{32b}
$$

$$u_{z3} = -\left[\frac{\partial}{\partial t}\left(\frac{P_{z3}}{\rho_o}\right) + u_o \frac{\partial}{\partial z}\left(\frac{P_{z3}}{\rho_o}\right) + u_{z1}\frac{\partial}{\partial z}\left(\frac{P_{z2}}{\rho_o}\right) + u_{z2}\frac{\partial}{\partial z}\left(\frac{P_{z1}}{\rho_o}\right)\right]$$

(32c)

where $\frac{P_z}{\rho_o}$ is the polarization displacement.

The remainder of this section will be used to describe a general method of solving Equation (31) and the solution through the third harmonic will be presented. First of all, Equation (31a) must be solved for a homogeneous first-harmonic solution. To obtain a solution for P_{z1} satisfying the Equation (31) and boundary conditions, consider the second total derivative with respect to time of a function of time and space, in operator form, as

$$\frac{d^2}{dt^2} = \frac{\partial^2}{\partial t^2} + 2u\frac{\partial^2}{\partial z\,\partial t} + u^2\frac{\partial^2}{\partial z^2} \quad . \tag{33}$$

Letting $u \approx u_o$ for the first harmonic solution, Equation (31a) can be written as

$$\frac{1}{u_o^2}\frac{d^2}{dt^2}P_{z1} + \beta_{q1}^2 P_{z1} = 0 \quad . \tag{34}$$

Equation (34) is in the form of an undamped oscillator equation, and an assumed solution, which will be shown to satisfy all boundary conditions, is in the form of

$$P_{z1} = F_1(z)\cos(\omega t - \beta_e z) \tag{35a}$$

where

$$F_1(z) = \frac{J_o}{\omega_{q1}}\,\frac{a}{2}\,\sin\beta_{q1} z \quad . \tag{35b}$$

When one uses the time-varying argument in the form of $(\omega t - \beta_e z)$ for either sin or cos time variations, one simplification is

$$\frac{d^n}{dt^n}P_{z1} = \frac{d^n}{dz^n}F_1(z) \tag{35c}$$

for $n = 1, 2, \ldots$. Equation (35) must satisfy two initial conditions:

$$\text{(1)}\quad P_{z1}\Big|_{t,\, z=0} = 0$$

$$\text{(2)}\quad u_{z1}\Big|_{t,\, z=0} = \frac{a}{2} \quad . \tag{36}$$

Next, Equation (31b) is solved most easily by substituting P_{z1} in the form of Equation (35a) into the drive terms of Equation (31b), simplifying them into the form of

$$G_o(z) + G_1(z) \cos 2(\omega t - \beta_e z) + G_2(z) \sin 2(\omega t - \beta_e z) \quad , \tag{37a}$$

where the functions $G(z)$ are in the general form of

$$a_o + a_2 \cos 2\beta_{q1} z + b_2 \sin 2\beta_{q1} z \quad ; \tag{37b}$$

where a_o, a_2, and b_2 are constants; that is, functions of the second space harmonic of the fundamental. Since the drive terms are in the form of Equation (37a), the solution for P_{z2} must be in the following form:

$$P_{z2} = g_o(z) + g_1(z) \cos 2(\omega t - \beta_e z) + g_2(z) \sin 2(\omega t - \beta_e z) \quad . \tag{38}$$

Substituting Equations (37) and (38) into Equation (31b) gives the solution for P_{z2} in terms of the drive terms as

$$\frac{d^2}{dz^2} g_m^2(z) + \beta_{q2}^2 g_m(z) = a_m + b_m \cos 2\beta_{q1} z + c_m \sin 2\beta_{q1} z \quad , \tag{39}$$

where a_m, b_m, and c_m are constants, and $m = 1, 2,$ or 3. Equation (39) is an undamped oscillator equation with drive terms in the form of constants, sines or cosines. The solution to this type of equation,[5] for the case where $\beta_{q2} \neq 2\beta_{q1}$, can be written as

$$g_m = \frac{a_m}{\beta_{q2}^2} + \frac{b_m \cos 2\beta_{q1} z}{\beta_{q2}^2 - 4\beta_{q1}^2} + \frac{c_m \sin 2\beta_{q1} z}{\beta_{q2}^2 - 4\beta_{q1}^2} \quad . \tag{40}$$

Thus knowing the drive terms, one can write the particular solution of the second harmonic polarisation, P_{z2p} immediately.

The complementary component of P_{z2} is necessarily contained in $g_1(z)$, because of the form of P_{z1}, and is

$$g_1(z) = g_1'(z) + g_a \cos \beta_{q2} z + g_b \sin \beta_{q2} z \quad , \tag{41}$$

where g_a and g_b are two arbitrary constants determined respectively by the boundary conditions:

$$P_{z2}\Big|_{t,\, z=0} = 0 \quad \text{such that} \quad g_a = g_1(0) - g_1'(0) \quad ,$$

and

$$u_{z2}\Big|_{t,\, z=0} = \frac{a^2}{16} \quad \text{such that} \quad g_b + u_{z2p}\Big|_{t,\, z=0} = -\frac{a^2}{16} \quad .$$

Finally, P_{z1} and P_{z2} can be substituted into the drive terms for Equation (31c). Then following the method outlined above for solving P_{z2}, the third harmonic, P_{z3}, can be obtained. This will give correction terms to the fundamental in addition to the third harmonic terms.

In general, using the fundamental, homogeneous solution in this form, the higher harmonics can be obtained in consecutive order by using the following approach: (In this case the nth harmonic is used instead of the third harmonic to illustrate the general method.)

A. Simplify the drive terms of the next (nth) harmonic by substituting for the (lower-order) drive terms the general form in which all the polarisation terms can be written; that is,

$$P_{z1} = f_1(z) \cos(\omega t - \beta_e z) \quad ,$$

$$P_{z2} = g_o(z) + g_1(z) \cos 2(\omega t - \beta_e z) + g_2(z) \sin 2(\omega t - \beta_e z) \quad ,$$

$$P_{z(n-1)} = h_o(z) + h_1(z) \cos(n-1)(\omega t - \beta_e z) + h_2(z) \sin(n-1)(\omega t - \beta_e z) \quad . \tag{42}$$

The expression for each harmonic of P_z contains only the largest magnitude terms; actually each odd harmonic P_z contains all lower odd harmonics, and each even harmonic P_z contains all lower even harmonics and a d-c term. In any case, substituting Equation (42) into the drive terms for P_{zn} harmonic reduces the drive terms (main terms only) to

$$I_o(z) + I_1(z) \cos n\omega t + I_2(z) \sin n\omega t \quad , \tag{43a}$$

where every term in the coefficients $I_o(z)$, $I_1(z)$, and $I_2(z)$ must be in the form of

$$a_o + \sum_n a_n \cos n\beta_{qm} z + \sum_n b_n \sin n\beta_{qm} z \quad , \tag{43b}$$

where

$n = 0, 2, 4, \ldots$ for even P_{zn} ,

and

$n = 1, 3, 5, \ldots$ for odd P_{zn} .

In general β_{qm} equals all combinations of β_q's such as $\beta_{q1}, \beta_{q2}, \beta_{q3}, \ldots, \beta_{qn}$;

$$\beta_{q1} \pm \beta_{q2},\ \beta_{q1} \pm \beta_{q3},\ \ldots,\ \beta_{q1} \pm \beta_{q(n-1)}\ ;$$

$$\beta_{q2} \pm \beta_{q3},\ \beta_{q2} \pm \beta_{q4},\ \ldots,\ \beta_{q2} \pm \beta_{q(n-2)}\ ;$$

$$\beta_{q1} \pm \beta_{q2} \pm \beta_{q3},\ \beta_{q1} \pm \beta_{q2} \pm \beta_{q4},\ \ldots,\ \beta_{q1} \pm \beta_{q2} \pm \beta_{q(n-3)}\ ;$$

$$\vdots$$

$$\beta_{q1} \pm \beta_{q2} \pm \beta_{q3} \pm \cdots \pm \beta_{q(n-r)} \quad \text{where } n = (1 + 2 + 3 + \ldots + r) \quad .$$

However, the largest terms of the higher harmonics are dependent on the first harmonic, so generally only the β_{q1} variations in space are kept. The second largest terms are dependent on β_{q2} variations; the third largest terms are dependent on β_{q3} and $\beta_{q1} \pm \beta_{q2}$ variations, etc. The complete P_{zn} harmonic will contain n "orders" of magnitudes of terms.

B. Because the driving terms of the undamped oscillator equation are in the form of Equation (43a),the solution, as in Equation (42), will be

$$P_{zn} = i_o(z) + i_1(z) \cos n(\omega t - \beta_e z) + i_2(z) \sin n(\omega t - \beta_e z) \quad . \tag{44}$$

Substituting this into the oscillator equation shows that the time variations can be separated out in every case and the drive terms are always in the form of Equation (43b). The complete particular solution P_{znp} is obtained by superposition of the individual terms and can be written by inspection from the drive terms shown in Tables I, II, and III.

TABLE I $\beta_{qn}^2 \neq \frac{n^2}{m^2} \beta_{qm}^2$

Drive Term	a_o	$a_n \cos n\beta_{qm} z$	$b_n \sin n\beta_{qm} z$
Solution	$\frac{a_o}{\beta_{qn}^2}$	$\frac{a_n}{\beta_{qn}^2 - n^2\beta_{qm}^2} \cos n\beta_{qm} z$	$\frac{b_n}{\beta_{qn}^2 - n^2\beta_{qm}^2} \sin n\beta_{qm} z$

TABLE II $\beta_{qn}^2 = \frac{n^2}{m^2} \beta_{qm}^2$

Drive Term	$a_n \cos\beta_{qn} z$	$b_n \sin\beta_{qn} z$
Solution	$\frac{a_n}{2} \frac{z}{\beta_{qn}} \sin\beta_{qn} z$	$\frac{b_n}{2} \frac{z}{\beta_{qn}} \cos\beta_{qn} z$

TABLE III $\beta_{qm}^2 \neq \frac{n^2}{m^2} \beta_{qn}^2$ and $\beta_{qm}^2 = \frac{n^2}{m^2} \beta_{qn}^2$

Drive Term	$a_n z \cos\beta_{qm} z$	$b_n z \sin\beta_{qm} z$
Solution	$+\frac{\beta_{qm} a_n}{3\beta_{qm}^2 + \beta_{qn}^2} z^2 \sin\beta_{qm} z$	$-\frac{\beta_{qm} a_n}{3\beta_{qm}^2 + \beta_{qn}^2} z^2 \cos\beta_{qm} z$

Table I gives the solution for the case when the driving frequency is not equal to the natural resonance of the oscillator equation, and Table II gives the solution for the case when the driving is equal to the natural resonance of the oscillator equation so that the growth of the term is linearly increasing in space toward infinity.

C. Thus P_{znp} is of the form

$$P_{znp} = i_o(z) + i_{1p}(z) \cos n(\omega t - \beta_e z) + i_2(z) \sin n(\omega t - \beta_e z) \quad . \tag{45}$$

The arbitrary constants must be added to Equation (45) before the complete solution of Equation (44) is obtained. Because of the initial choice of the cosine time variation of P_{z1}, the two arbitrary coefficients are contained in the coefficient of the cosine term. In order that the initial conditions be satisfied, the coefficient of $i_1(z)$ in Equation (44) in terms of $i_{1p}(z)$ in Equation (45) and the arbitrary coefficients is

$$i_1(z) = i_{1p}(z) + i_{1a} \cos \beta_{qn} z + i_{1b} \sin \beta_{qn} z \quad , \tag{46}$$

where i_{1a} and i_{2b} are arbitrary constants. Using the knowledge that

$$P_{zn} = P_{znp} + P_{znc} \quad , \tag{47a}$$

and

$$u_{zn} = u_{znp} + u_{znc} \quad , \tag{47b}$$

where the subscripts p and c refer to the particular and complementary solution respectively, one can use the initial conditions to determine the arbitrary constants. The two initial conditions are:

(1) $P_z\Big|_{z=0} = 0$ for all time variations. Consequently

$$P_{zn}\Big|_{z=0} = 0$$

resulting in

$$i_{1a} = i_o(0) - i_{1p}(0) \tag{48a}$$

(2) From Webster's theory for the velocity [from Equation (16)]

$$\frac{u_z}{u_o}\Bigg|_{z,\, t=0} = \left(1 - \frac{a^2}{16}\right) + \left(\frac{a}{2} + \frac{3a^3}{64}\right) - \frac{a^2}{16} + \frac{a^3}{64} \dots .$$

Consequently $u_{zn}\Big|_{z,\, t=0}$ is the coefficient of the nth harmonic of the velocity, so that

$$i_{1b} = u_{zn}\Big|_{z,\, t=0} - \frac{1}{\beta_{qn}} \frac{d}{dt} i_{1p}(0) \quad . \tag{48b}$$

Comparing Equations (46) and (47) one can show that

$$\frac{d}{dt} i_{1p}(z) = \frac{d}{dt} P_{znp} \quad .$$

D. After the two arbitrary constants are obtained, P_{zn} can be written and $P_{z(n+1)}$ harmonic can be solved in the same manner. This successive approximation is straightforward, but becomes extremely lengthy for higher harmonics.

The solution for the polarization, P_z, through the third harmonic, keeping only the largest (highest order) terms is given in Equation (49). From Equations (32) and (49), velocity is given in Equation (50), and from Equations (21a) and (49), the current density is given in Equation (51). In Equations (49), (50) and (51), the following ratios are used:

$$\xi_{12} = \frac{\beta_{q1}}{\beta_{q2}} \quad ; \qquad \xi_{13} = \frac{\beta_{q1}}{\beta_{q3}} \quad ; \qquad \xi_{23} = \frac{\beta_{q2}}{\beta_{q3}} \quad .$$

The constants K_1 - K_{10} are defined in Appendix C.

The Equations (49), (50) and (51) are necessarily lengthy and complicated because of the harmonic variations and are more easily represented graphically. The harmonic content of the velocity and the current for a typical finite beam are shown in Figures 4 and 5. The typical beam case was done for the SAL-36 klystron.

$$
\begin{aligned}
P_z = & -J_o \frac{a}{Z} \frac{1}{\omega_{q1}} \sin\beta_{q1} z \cos(\omega t - \beta_e z) \\
& + J_o \left(\frac{a}{Z}\right)^2 \frac{\omega}{\omega_{q2}^2} \left\{ \left[\left(\frac{1}{4} - \frac{3}{4} \frac{1}{4\xi_{12}^2 - 1} \right) \cos 2\beta_{q1} z - \frac{\xi_{12}^2 - 1}{4\xi_{12}^2 - 1} \cos\beta_{q2} z \right] \sin 2(\omega t - \beta_e z) \right\} \\
& - J_o \left(\frac{a}{Z}\right)^3 \frac{\omega^2}{\omega_{q1}\,\omega_{q3}^2} \left\{ \left[K_1 \sin 3\beta_{q1} z + K_2 \sin\beta_{q1} z \right.\right. \\
& \left. + K_3 \sin(\beta_{q2} + \beta_{q1}) z + K_4 \sin(\beta_{q2} - \beta_{q1}) z + K_5 \sin\beta_{q3} z \right] \cos 3(\omega t - \beta_e z) \\
& + \left[K_6 \sin 3\beta_{q1} z + K_7 \sin\beta_{q1} z \right. \\
& \left.\left. + K_8 \sin(\beta_{q2} + \beta_{q1}) z + K_9 \sin(\beta_{q2} - \beta_{q1}) z + K_{10} \sin\beta_{q3} z \right] \cos(\omega t - \beta_e z) \right\}
\end{aligned}
\tag{49}
$$

$$
\begin{aligned}
\frac{u_z}{u_o} = {} & \frac{a}{2}\cos\beta_{q1}z\cos(\omega t-\beta_e z) - \left(\frac{a}{2}\right)^2 \frac{\omega}{\omega_{q2}} \left\{\left[-\left(\frac{3}{2}\,\frac{\xi_{12}}{4\xi_{12}^{2}-1} - \frac{1}{\xi_{12}}\right)\sin 2\beta_{q1}z\right.\right. \\
& \left.\left. + \frac{\xi_{12}^{2}-1}{4\xi_{12}^{2}-1}\sin\beta_{q2}z\right]\sin 2(\omega t-\beta_e z)\right\} + \left(\frac{a}{2}\right)^3\left(\frac{\omega}{\omega_{q3}}\right)^2 \left\{\left[3K_1\cos 3\beta_{q1}z + K_2\cos\beta_{q1}z\right.\right. \\
& \left. + K_3\left(\frac{1}{\xi_{12}}+1\right)\cos(\beta_{q2}+\beta_{q1})z + K_4\left(\frac{1}{\xi_{12}}-1\right)\cos(\beta_{q2}-\beta_{q1})z + K_5\frac{1}{\xi_{13}}\cos\beta_{q3}z\right]\cos 3(\omega t-\beta_e z) \\
& + \left[3K_6\cos 3\beta_{q1}z + K_7\cos\beta_{q1}z + K_8\left(\frac{1}{\xi_{12}}+1\right)\cos(\beta_{q2}+\beta_{q1})z + K_9\left(\frac{1}{\xi_{12}}-1\right)\cos(\beta_{q2}-\beta_{q1})z\right. \\
& \left. + K_{10}\frac{1}{\xi_{13}}\cos\beta_{q3}z\right]\cos(\omega t-\beta_e z) + \frac{1}{\xi_{13}^{2}}\left[\left(\frac{3}{8}\,\frac{\xi_{12}^{2}}{4\xi_{12}^{2}-1} - \frac{1}{4}\,\frac{1}{\xi_{12}}\right)(\cos\beta_{q1}z - \cos 3\beta_{q1}z)\right. \\
& \left. + \frac{1}{4}\left(\frac{\xi_{12}^{2}-1}{4\xi_{12}^{2}-1}\right)\cos(\beta_{q2}-\beta_{q1})z - \cos(\beta_{q2}+\beta_{q1})\right]\left(\cos 3(\omega t-\beta_e z) - \cos(\omega t-\beta_e z)\right) \\
& + \frac{1}{\xi_{13}^{2}}\left[\left(\frac{1}{8} - \frac{3}{8}\,\frac{1}{4\xi_{12}^{2}-1}\right)\cos\beta_{q1}z - \frac{3}{2}\,\frac{1}{4\xi_{12}^{2}-1}\cos 3\beta_{q1}z\right. \\
& \left.\left. - \frac{1}{2}\,\frac{\xi_{12}^{2}-1}{4\xi_{12}^{2}-1}\left(\cos(\beta_{q2}+\beta_{q1})z + \cos(\beta_{q2}-\beta_{q1})z\right)\right]\left[\cos 3(\omega t-\beta_e z) + \cos(\omega t-\beta_e z)\right]\right\}
\end{aligned}
\qquad (50)
$$

$$
\begin{aligned}
\frac{J_z}{J_o} = &+ \frac{a}{2}\,\frac{\omega}{\omega_{q1}}\,\sin\beta_{q1}z\,\sin(\omega t - \beta_e z) \\
&+ \left(\frac{a}{2}\right)^2 \frac{\omega^2}{\omega_{q2}^{\,2}} \left\{ 2\left[\frac{1}{4} - \frac{3}{4}\,\frac{1}{4\xi_{12}^{\,2} - 1}\,\cos 2\beta_{q1}z - \frac{\xi_{12}^{\,2} - 1}{4\xi_{12}^{\,2} - 1}\,\cos\beta_{q2}z\right] \cos 2(\omega t - \beta_e z) \right\} \\
&+ \left(\frac{a}{2}\right)^3 \frac{\omega^3}{\omega_{q1}\,\omega_{q3}^{\,2}} \Bigg\{ 3\Big[K_1 \sin 3\beta_{q1}z + K_2 \sin\beta_{q1}z + K_3 \sin(\beta_{q2} + \beta_{q1})z \\
&\qquad + K_4 \sin(\beta_{q2} - \beta_{q1})z + K_5 \sin\beta_{q3}z\Big] \sin 3(\omega t - \beta_e z) + \Big[K_6 \sin 3\beta_{q1}z \\
&\qquad + K_7 \sin\beta_{q1}z + K_8 \sin(\beta_{q2} + \beta_{q1})z + K_9 \sin(\beta_{q2} - \beta_{q1})z \\
&\qquad + K_{10} \sin\beta_{q3}z\Big] \sin(\omega t - \beta_e z)\Bigg\}
\end{aligned} \tag{51}
$$

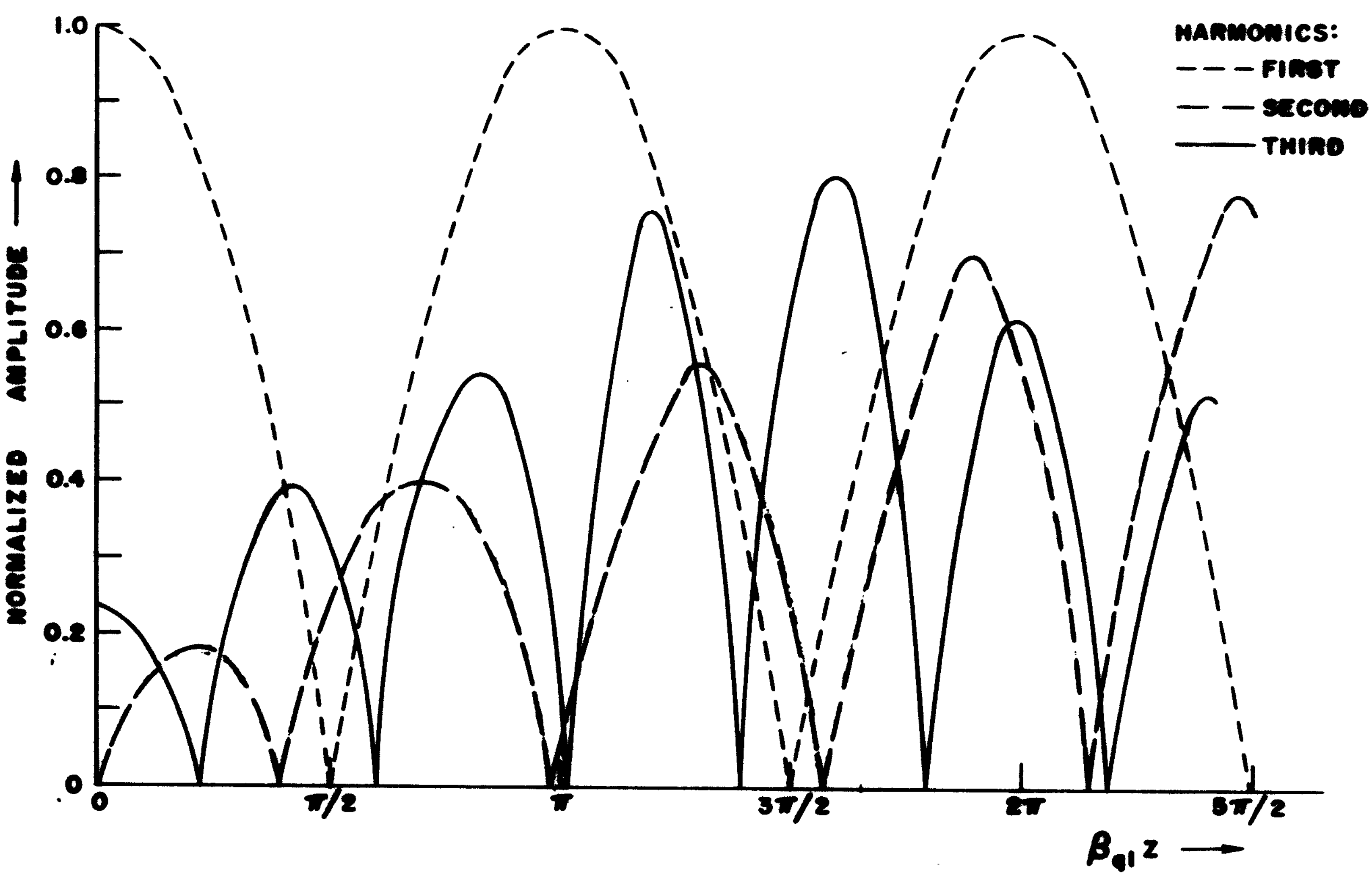

Figure 4. Velocity Harmonics versus $\beta_{ql} z$ for Finite Beam.

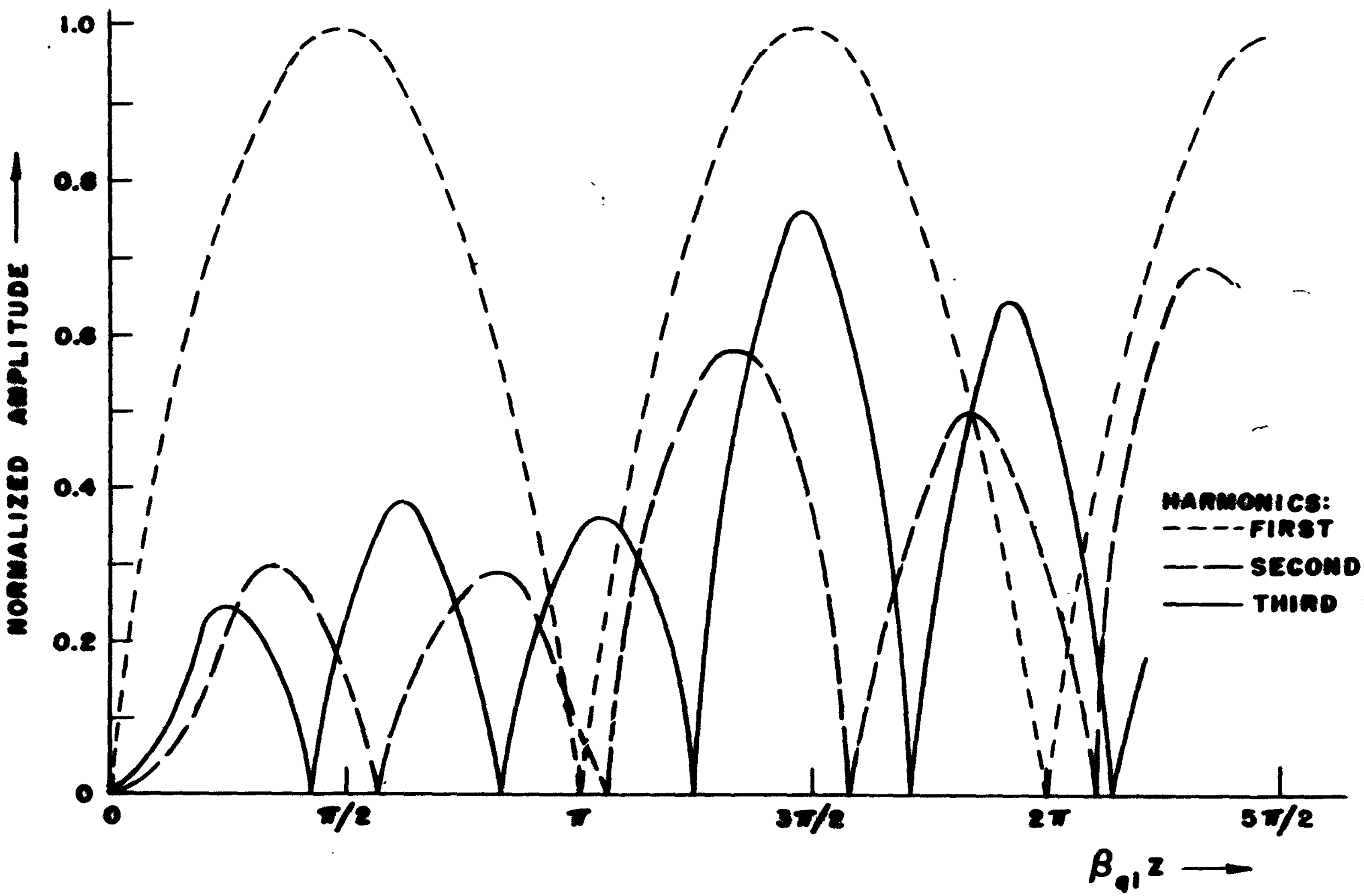

Figure 5. Current Harmonics versus $\beta_{ql} z$ for Finite Beam.

Two special cases of the polarization, velocity, and current given in the last three equations are interesting to examine. The first is the special case of the infinite beam (or infinite frequency); that is, the space-charge reduction factor for all space-charge harmonics becomes unity. Consequently, $\xi_{12} = \xi_{13} = \xi_{23} = 1$, and the polarization velocity and current reduce to the following:

$$
\begin{aligned}
P_z = & - J_o \frac{a}{2} \frac{1}{\omega_p} \sin\beta_p z \cos(\omega t - \beta_e z) \\
& + \frac{J_o}{4} \left(\frac{a}{2}\right)^2 \frac{\omega}{\omega_p^2} \left\{ \left[1 - \cos 2\beta_p z\right] \sin 2(\omega t - \beta_e z \right\} \\
& + \frac{J_o}{32} \left(\frac{a}{2}\right)^3 \frac{\omega^2}{\omega_p^3} \left\{ 3 \left[- \sin 3\beta_p z + 3 \sin\beta_p z\right] \cos 3(\omega t - \beta_e z) \right. \\
& \left. + \left[- \sin 3\beta_p z + 3 \sin\beta_p z\right] \cos(\omega t - \beta_e z) \right\}
\end{aligned}
\tag{52}
$$

$$
\begin{aligned}
\frac{u_z}{u_o} = & \ 1 + \left(\frac{a}{2}\right) \cos\beta_p z \cos(\omega t - \beta_e z) \\
& - \frac{1}{4} \left(\frac{a}{2}\right)^2 \frac{\omega}{\omega_p} \sin 2\beta_p z \sin 2(\omega t - \beta_e z) \\
& + \frac{1}{32} \left(\frac{a}{2}\right)^2 \left(\frac{\omega}{\omega_p}\right)^2 \left\{ - 5 \left[- \cos 3\beta_p z + \cos\beta_p z\right] \cos 3(\omega t - \beta_e z) \right. \\
& \left. + 7 \left[- \cos 3\beta_p z + \cos\beta_p z\right] \cos(\omega t - \beta_e z) \right\}
\end{aligned}
\tag{53}
$$

$$\frac{J_z}{J_o} = -\frac{a}{2}\frac{1}{\omega_p}\sin\beta_p z \sin(\omega t-\beta_e z) - \frac{1}{2}\left(\frac{a}{2}\right)^2\left(\frac{\omega}{\omega_p}\right)^2 \left\{\left[1-\cos 2\beta_p z\right]\cos 2(\omega t-\beta_e z)\right\}$$
$$+\frac{1}{32}\left(\frac{a}{2}\right)^3\left(\frac{\omega}{\omega_p}\right)^3\left\{9\left[-\sin 3\beta_p z + 3\sin\beta_p z\right]\sin 3(\omega t-\beta_e z) + \left[-\sin 3\beta_p z + 3\sin\beta_p z\right]\sin(\omega t-\beta_e z)\right\} \tag{54}$$

Equations (52), (53) and (54) can be further reduced to Webster's velocity and current series.[1,2]

The other special case is the opposite situation which is that of an extremely thin beam (or low frequency); that is, the space-reduction factor, while having an extremely small value, is directly proportional to the order of the space-charge harmonics. Consequently, $\xi_{12} = \frac{1}{2}$, $\xi_{13} = \frac{1}{3}$, and $\xi_{23} = \frac{2}{3}$, and the solutions in Tables II and III must be used. The polarization, velocity, and current density are rapidly growing functions in space and reduce to the following:

$$P_z = -J_o\frac{a}{2}\frac{1}{\omega_{q1}}\sin\beta_{q1}z\cos(\omega t-\beta_e z)$$
$$+J_o\left(\frac{a}{2}\right)^2\left(\frac{\omega}{\omega_{q1}^2}\right)\left\{\frac{3}{16}\beta_{q1}z\sin 2\beta_{q1}z\sin 2(\omega t-\beta_e z)\right\}$$
$$-J_o\left(\frac{a}{2}\right)^3\left(\frac{\omega^2}{\omega_{q1}^2}\right)\left\{\left[\left(-\frac{1}{2}+\frac{5}{16}\beta_{q1}^2 z^2\right)\sin 3\beta_{q1}z + \left(-\frac{5}{2}-\frac{5}{16}\beta_{q1}^2 z^2\right)\sin\beta_{q1}z\right]\cos 3(\omega t-\beta_e z)\right.$$
$$\left.+\left[\left(-\frac{1}{2}+\frac{1}{16}\beta_{q1}^2 z^2\right)\sin 3\beta_{q1}z + \left(+\frac{3}{2}-\frac{1}{16}\beta_{q1}^2 z^2\right)\sin\beta_{q1}z\right]\cos(\omega t-\beta_e z)\right\} \tag{55}$$

$$\frac{u_z}{u_o} = 1 + \frac{a}{2} \cos\beta_{q1}z \; \cos 2(\omega t - \beta_e z)$$

$$+\left(\frac{a}{2}\right)^2 \frac{\omega}{\omega_{q1}} \left[\frac{3}{8} \beta_{q1}z \; \cos 2\beta_{q1}z \; \sin 2(\omega t - \beta_e z)\right]$$

$$-\left(\frac{a}{2}\right)^3 \left(\frac{\omega}{\omega_{q1}}\right)^2 \left\{\left[\left(-\frac{3}{2} + \frac{15}{16}\beta_{q1}{}^2 z^2\right) \cos 3\beta_{q1}z \right.\right.$$

$$\left. + \left(-\frac{5}{2} - \frac{5}{16}\beta_{q1}{}^2 z^2\right) \cos\beta_{q1}z\right] \cos 3(\omega t - \beta_e z)$$

$$+ \left[\left(-\frac{3}{2} + \frac{3}{16}\beta_{q1}{}^2 z^2\right) \cos 3\beta_{q1}z \right.$$

$$\left.\left. + \left(-\frac{3}{2} - \frac{1}{16}\beta_{q1}{}^2 z^2\right) \cos\beta_{q1}z\right] \cos(\omega t - \beta_e z)\right\} \quad (56)$$

$$\frac{J}{J_o} = \frac{a}{2} \frac{\omega}{\omega_{q1}} \sin\beta_{q1}z \; \sin(\omega t - \beta_e z)$$

$$+\left(\frac{a}{2}\right)^2 \left(\frac{\omega}{\omega_{q1}}\right)^2 \left[\frac{3}{8} \beta_{q1}z \; \sin 2\beta_{q1}z \; \cos 2(\omega t - \beta_e z)\right]$$

$$+\left(\frac{a}{2}\right)^3 \left(\frac{\omega}{\omega_{q1}}\right)^3 \left\{\left[\left(-\frac{3}{2} + \frac{15}{16}\beta_{q1}{}^2 z^2\right) \sin 3\beta_{q1}z \right.\right.$$

$$\left. + \left(-\frac{15}{2} - \frac{15}{16}\beta_{q1}{}^2 z^2\right) \sin\beta_{q1}z\right] \sin 3(\omega t - \beta_e z)$$

$$+ \left[\left(-\frac{1}{2} + \frac{1}{16}\beta_{q1}{}^2 z^2\right) \sin 3\beta_{q1}z \right.$$

$$\left.\left. + \left(+\frac{3}{2} - \frac{1}{16}\beta_{q1}{}^2 z^2\right) \sin\beta_{q1}z\right] \sin(\omega t - \beta_e z)\right\} \quad (57)$$

Equations (52) through (54) and (55) through (57) are also lengthy because of the harmonic variations. Consequently, Figures 6-9 are presented showing the harmonic content of the velocity and current for the infinite beam and the thin beam cases.

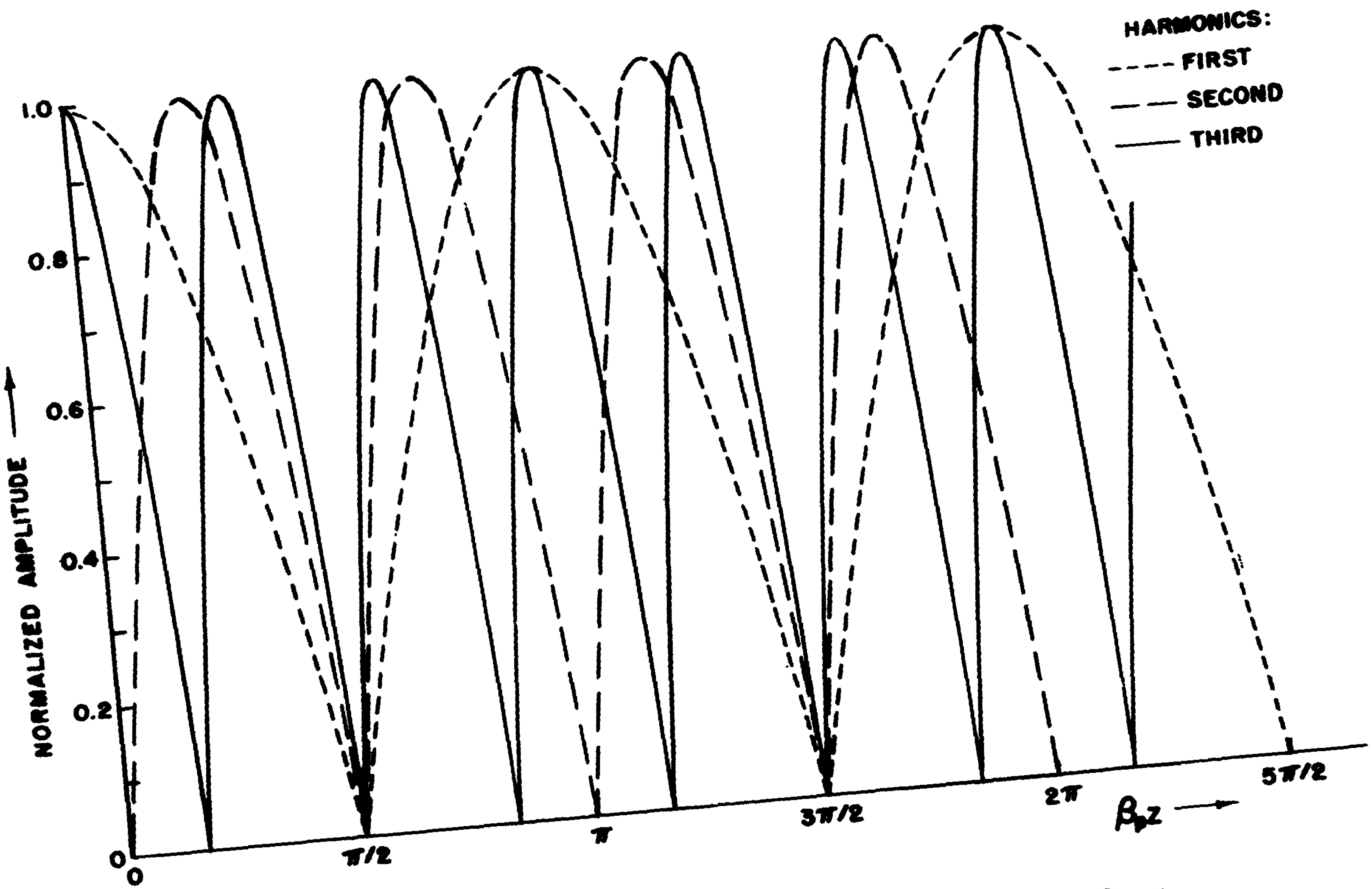

Figure 6. Velocity Harmonics versus $\beta_p z$ for Infinite Beam.

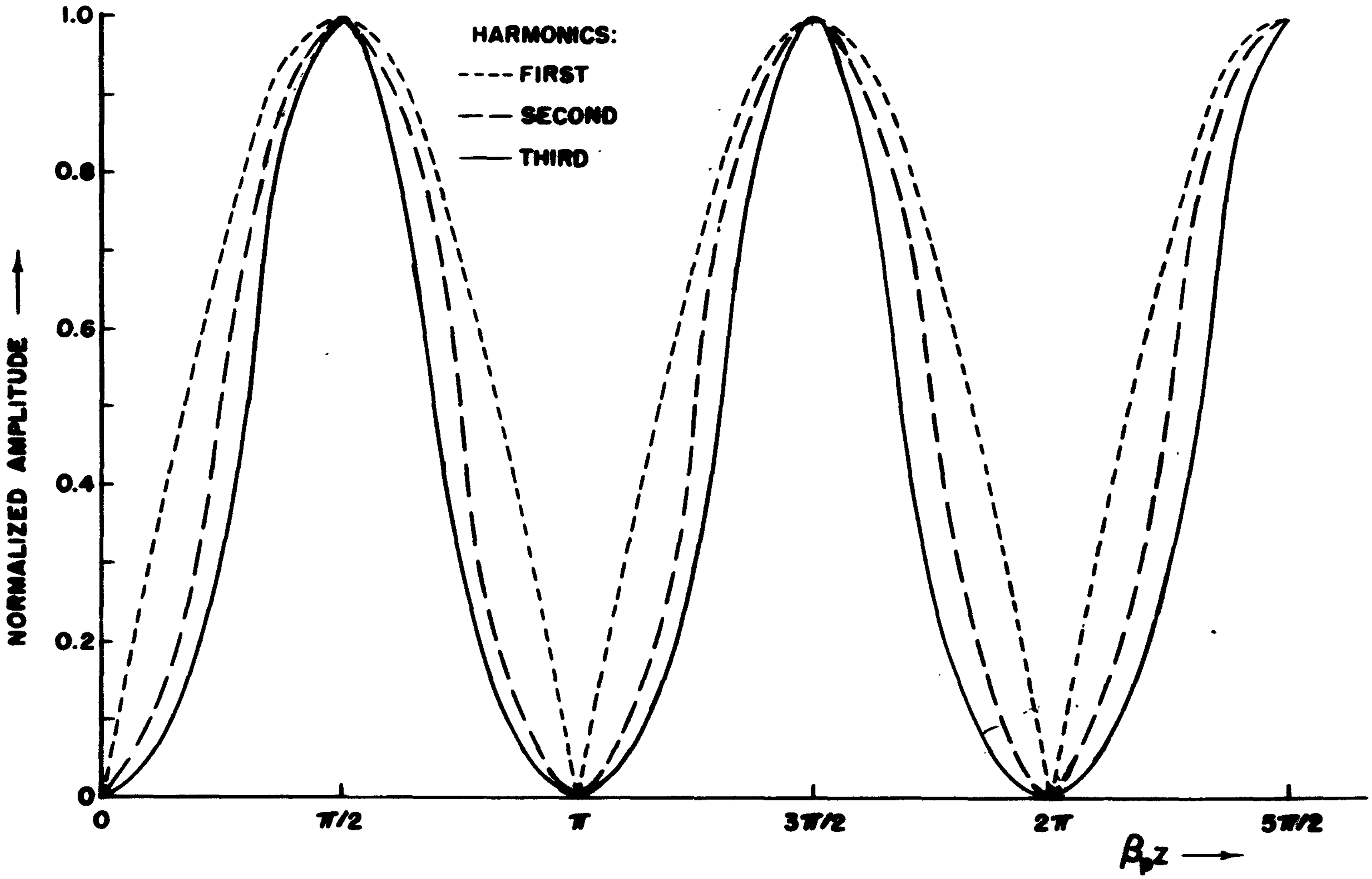

Figure 7. Current Harmonics versus $\beta_p z$ for Infinite Beam.

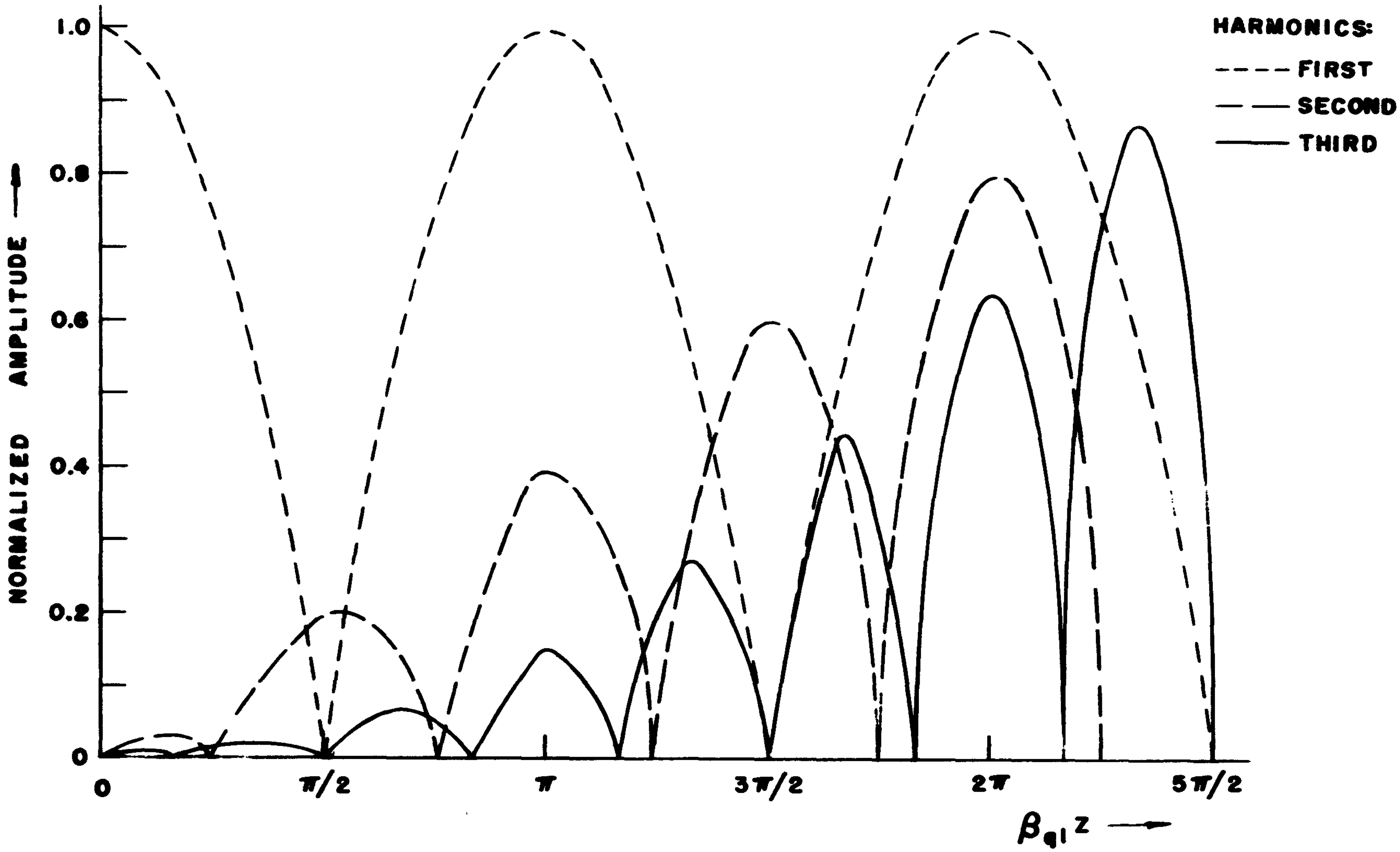

Figure 8. Velocity Harmonics versus β_{ql} z for Thin Beam.

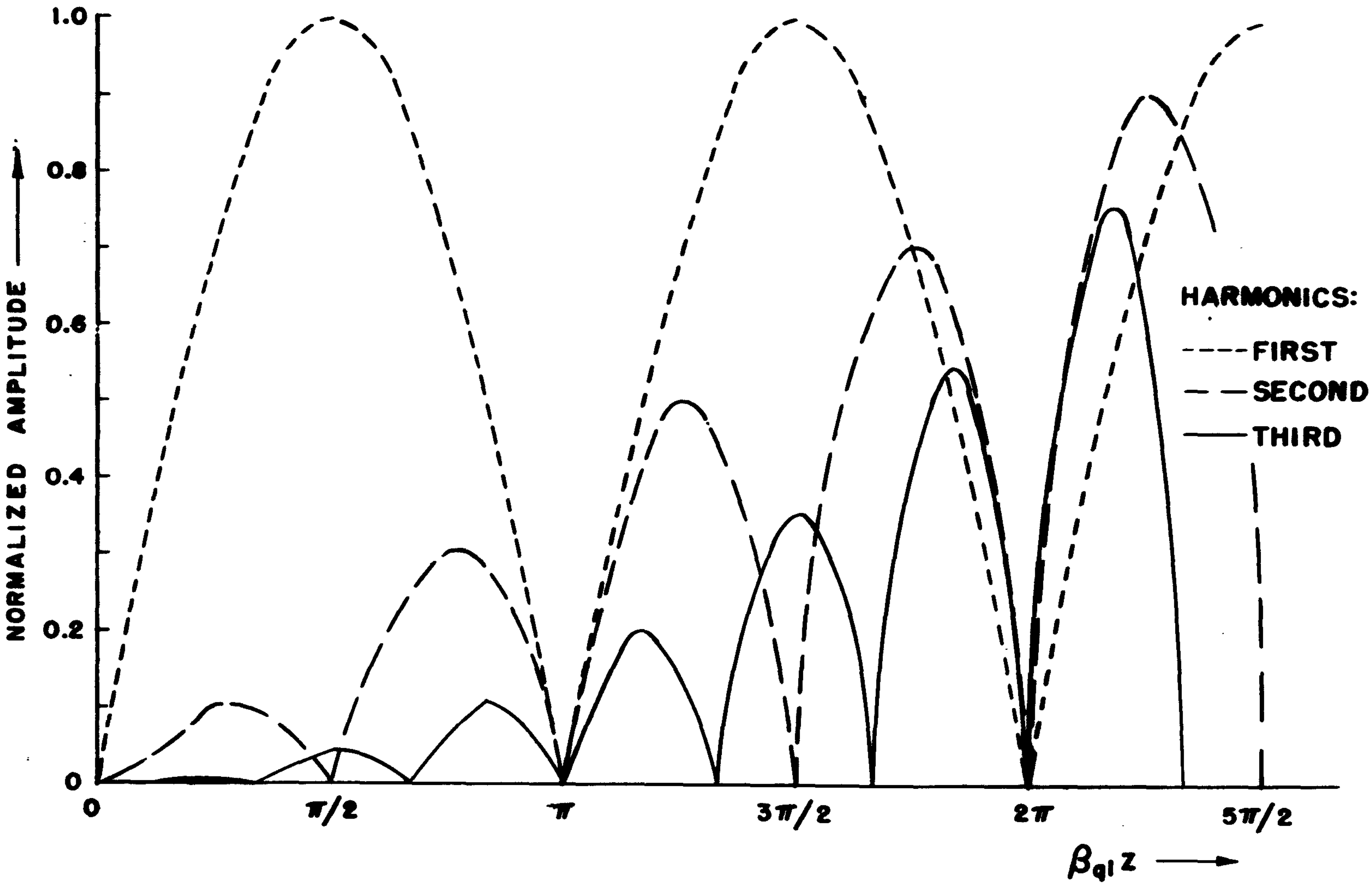

Figure 9. Current Harmonics versus $\beta_{ql} z$ for Thin Beam.

III. CORRECTION TO SPACE-CHARGE PHASE CONSTANTS

Additional Symbols Used in this Section

ζ_{mn} Interaction impedance; it is the ratio of the electric field at point m caused by the current at point n .

Δ_n Normalized space-charge wave correction for the nth space-charge wave harmonic.

In the previous section, the space-charge wave variations in distance were obtained by solving a linear, undamped oscillator equation (Equation 31). This produced a fast and a slow longitudinal wave whose phase constants, $\beta_e \pm \beta_{qn}$, are slightly larger or smaller than the fast-varying electron beam phase constant, β_e . Consequently the beating of the fast and the slow waves produced a (fundamental) solution of the polarization and current, which was shown respectively to be

$$P_{z1} = \frac{J_o}{\omega_{q1}} \left(\frac{a}{2} \right) \sin \beta_{q1} z \cos (\omega t - \beta_e z) \quad , \tag{58}$$

$$J_{z1} = J_o \frac{\omega}{\omega_{q1}} \left(\frac{a}{2} \right) \sin \beta_{q1} \, z \sin (\omega t - \beta_e z) \quad . \tag{59}$$

From Equations (19) and (28) the longitudinal electric field can be written as

$$E_{z1} = \frac{1}{\epsilon_o} \frac{J_o}{\omega_{q1}} \left(\frac{a}{2} \right) \sin \beta_{q1} z \ \cos (\omega t - \beta_e z) \quad . \tag{60}$$

The ratio of the electric field to the current for the corresponding

$e^{j(\omega t - \beta_e z)}$ terms of either the fast or slow space-charge waves, is an impedance, ζ, and from Equations (59) and (60) is written as

$$\zeta = \frac{E_{z1}}{J_{z1}} = \frac{j}{\epsilon_0 \omega} \quad . \tag{61}$$

This shows clearly that the impedance does not depend on whether the fast or slow space-charge wave is being considered. This is approximately correct, but is not true. In the remainder of this section, the correct expression for the impedances, dependent on the phase constants of the two space-charge waves, will be shown. This work is based on an article by McIsaac and Wang[6].

In fundamental electromagnetic theory, the electromagnetic fields can be expressed in terms of a certain distribution of current sources. If a wave guide has simple boundary conditions, as for example, a cylindrical wave guide, the total electric field at a point, m, is a summation by superposition of the electric fields produced by an individual exciting current ray in Figure 10. Consequently, if the current

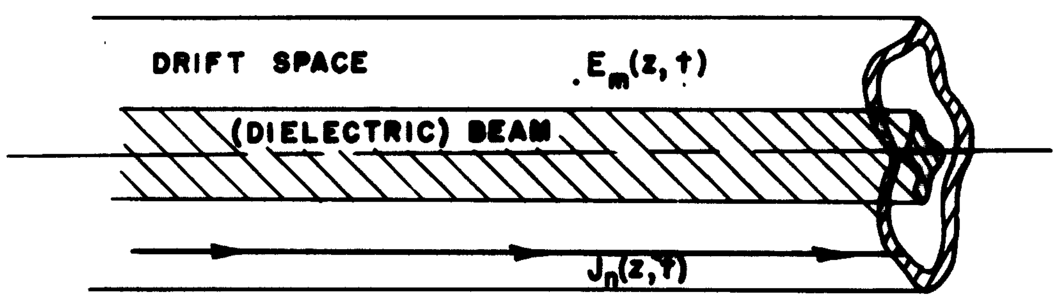

Figure 10. Electric Field and Current Representation in Drift Space.

distribution in the wave guide can be represented by a summation of current rays, or delta functions, the total electric field, at a point, m , is the integration of the products of each delta current and its response function.

The integration for this interaction theory of current and electric field is, in general, complicated, but the specific case of a drift space is relatively easy. It will be assumed that the cross section of the drift tube is constant in the longitudinal direction, and the beam can be considered as a dielectric in the drift tube. Both the electric fields and the currents will be varying exponentially in time and space. The circular frequency, ω , is constant, and the phase velocity of the currents must be very close to the phase velocity, β_e , of the electron beam, since strong interactions exist among the currents. Let the total phase constant, β' , equal the electron phase velocity plus a correction term, β , such that

$$J_n'(z,t) = J_{on} e^{j(\omega t - \beta' z)} = J_n e^{j(\omega t - \beta_e z)} \quad , \tag{62a}$$

and

$$E_m'(z,t) = E_{om} e^{j(\omega t - \beta' z)} = E_m e^{j(\omega t - \beta_e z)} \quad , \tag{62b}$$

where J_n and E_m are slowly varying functions of z such that

$$J_n = e^{-j\beta z} \quad , \tag{63a}$$

and

$$E_m = e^{-j\beta z} \quad , \tag{63b}$$

where

$$\beta' = \beta_e + \beta \text{ and } \beta \ll \beta_e \quad .$$

A differential equation describing the interaction theory can now be written and solved revealing a more accurate value for the special variations of the electric field and current. In order that Equation (62) satisfy Maxwell's equations, the electric field at point m caused by the current at point n must be directly related as follows:

$$E_m = j\,\zeta_{mn}(\beta', \omega)\, J_n \quad , \tag{64}$$

where the fast variation $e^{j(\omega t - \beta_e z)}$ has been canceled. The important detail in Equation (64) is that the impedance, $\zeta_{mn}(\beta', \omega)$ is a function of β' , the total phase constant, not just β_e . This means that points m and n must be relatively close to each other because $\zeta_{mn}(\beta', \omega)$ varies in space. However, the closer the points are together, the more accurate Equation (64) becomes, when a single value of $\zeta_{mn}(\beta' \omega)$ is used.

For the most convenient as well as most accurate form of Equation (64) , the point m will be the same as point n . Thus the electric field will be evaluated at the point at which the currents act, and the impedance becomes a driving-point impedance. Thus Equation (64) becomes:

$$E_n = j\,\zeta_{nn}(\beta', \omega)\, J_n \quad . \tag{65}$$

The slow variations of the impedance can be more easily studied if separated from the fast variations. This can be accomplished by expanding the impedance in a Taylor series around β_e . By using only the first two terms of the exapnsion, Equation (65) becomes

$$E_n = j\left[\zeta_{nn}(\beta_e, \omega) + j\,\frac{\partial\,\zeta_{nn}(\beta_e, \omega)}{\partial\,\beta_e}\,\beta\right] J_n \quad . \tag{66}$$

Taking the derivative of the current from Equation (63a), we can write the slow-varying phase constant β as

$$\beta = - \frac{dJ_n}{dz} \frac{1}{J_n} \quad . \tag{67}$$

Substituting Equation (67) into Equation (66), eliminates the slowly varying β, giving the following differential equation:

$$E_n = j\zeta_{nn}(\beta_e, \omega) J_n - \frac{\partial \zeta_{nn}(\beta_e, \omega)}{\partial \beta_e} \frac{dJ_n}{dz} \quad . \tag{68}$$

From the continuity equation in Equation (21a), the slow-varying polarization, P_n and normalized current density are related as follows:

$$\frac{J_n}{J_o} = j\beta_e \left(\frac{P_n}{\rho_o}\right) \quad . \tag{69}$$

Substituting Equation (69) into Equation (68) yields

$$\frac{E_n}{J_o} = - \zeta_{nn} \beta_e \left(\frac{P_n}{\rho_o}\right) - j\beta_e \frac{\partial \zeta_{nn}}{\partial \beta_e} \frac{d}{dz} \left(\frac{P_n}{\rho_o}\right) \quad . \tag{70}$$

From Maxwell's equation,

$$\nabla X H = J_c + \frac{\partial D}{\partial t} \quad , \tag{71}$$

the maximum impedance occurs when $\nabla X H = 0$ and is

$$\zeta_{nn\,(max)} = \frac{1}{\omega \epsilon_o} \quad . \tag{72}$$

The ratio of impedance at point n to the maximum possible impedance at

n is then $\zeta_{nn} \omega \epsilon_o$. This is also the ratio of the space-charge field to the maximum possible field at the point n, which was defined in the previous section in Equation (27) as the space-charge reduction factor. The impedance, in other words, also describes the decrease in the space-charge electric field caused by fringing and is related to the reduced plasma frequency as

$$\zeta_{nn} \omega \epsilon_o = R^2 = \frac{\omega_q^2}{\omega_p^2} \quad . \tag{73}$$

From the force equation, the ratio of field to d-c current density can be written as

$$\frac{E_n}{J_o} = \frac{u_o^2}{J_o \eta} \frac{du}{dz} = \frac{\omega}{\omega_p^2 \epsilon_o} \frac{dw}{dz} \quad , \tag{74}$$

where

$$u = u_o e^{j(\omega t - \beta' z)} = u_n e^{j(\omega t - \beta_e z)} \quad ,$$

$$u_n = u_o e^{-j\beta z} \quad ,$$

and

$$w = \frac{u_n}{u_o} \quad .$$

If Equations (73) and (74) are substituted into Equation (70), the impedance is eliminated, and the interaction equation can be written as

$$\frac{dw}{dz} = -\left(\frac{\beta_q}{\beta_e}\right)^2 \left(\frac{P_n}{\rho_o}\right) + j\beta_e \frac{\partial}{\partial \beta_e} \left(\frac{\beta_q}{\beta_e}\right)^2 \frac{d}{dz} \left(\frac{P_n}{\rho_o}\right) \quad . \tag{75}$$

Using the simplification made in Equation (35c), gives the relationship between the normalized slow-varying velocity and the displacement as

$$w = \frac{d}{dz} \quad \frac{P_n}{\rho_o} \quad . \tag{76}$$

Finally, substituting Equation (76) into Equation (75), one can write the interaction equation as a function of the displacement as

$$\frac{d^2}{dz^2}\left(\frac{P_n}{\rho_o}\right) + j\beta_e \ \frac{\partial}{\partial \beta_e}\left(\frac{\beta_q}{\beta_e}\right)^2 \frac{d}{dz}\left(\frac{P_n}{\rho_o}\right) + \left(\frac{\beta_q}{\beta_e}\right)^2 \left(\frac{P_n}{\rho_o}\right) = 0 \quad . \tag{77}$$

Equation (77) is an undamped oscillator equation very similar to Equation (31) , but differs in that it has an additional phase-shift term proportional to the first derivative of the polarization, which is the velocity. This imaginary phase-shift term can be used to represent the energy required to supply the Poynting power flow associated with a bunched beam in the velocity modulated klystron.

Using any of the common methods[5] for solving the oscillator equation, Equation (77),one can write two solutions as

$$A_{1,2} = j\,\frac{\beta_{qn}}{\beta_e}\left(+\frac{1}{2}\,C\,\frac{\beta_{qn}}{\beta_e} \pm \sqrt{1 + \frac{1}{4}\,C^2\left(\frac{\beta_{qn}}{\beta_e}\right)^2}\right) , \tag{78}$$

where

$$C = -\frac{\partial \ln\left(\frac{\beta_{qn}}{\beta_e}\right)^2}{\partial \ln \beta_e} \quad .$$

The subscript, n , is the number of the space-charge wave harmonic.

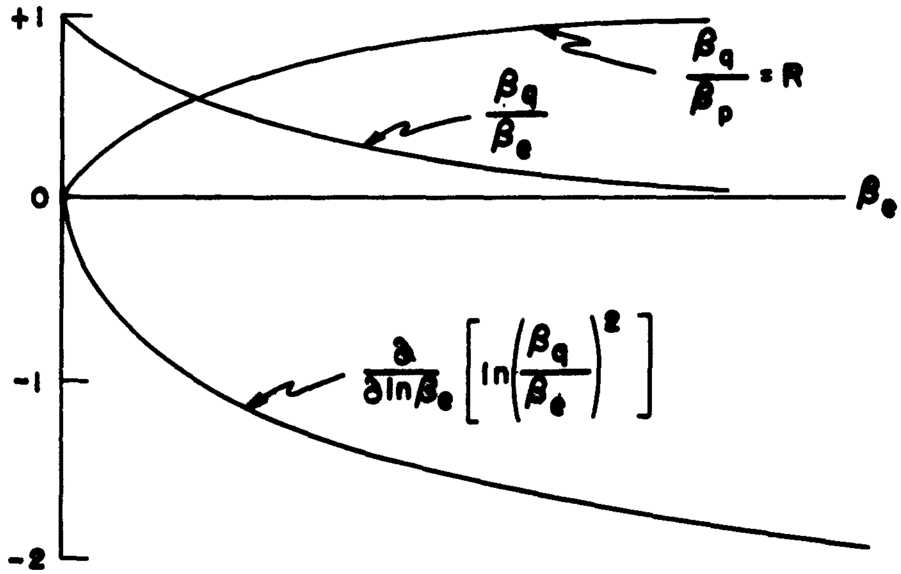

Figure 11. Variations of Components of $\Delta_{1,2}$ versus β_e.

In order to see how the $\Delta_{1,2}$ solutions vary with β_e the curves in Figures 11 and 12 are helpful.

From Figure 11, it is clear that C is always a positive quantity. Consequently, the quantity $\frac{1}{2} C \frac{\beta_{qn}}{\beta_e} = \Delta_n$ in Equation (78) can be drawn from Figure 11 to be as illustrated in Figure 12. Thus Figure 12 shows that Δ_n reaches a maximum at about $\beta_e = 1000$ radians per meter and is zero at zero frequency and at infinite frequency. The energy required to supply the Poynting power flow is proportional to Δ_n.

The solution, Equation (78), can be written in terms of a correction to the space-charge wave harmonic as

$$\beta_{qn}\pm = \beta_{qn}\left(\pm\sqrt{1 + \Delta_n^2} + \Delta_n\right) , \qquad (79)$$

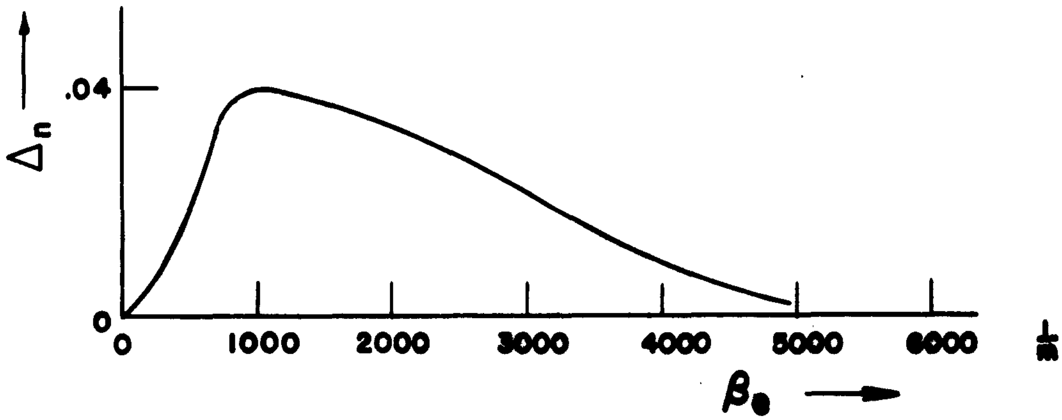

Figure 12. Δ_n versus β_e.

where plus and minus refer to the slow and fast wave respectively. The phase velocity, v_p, is written as

$$v_p = \frac{\omega}{\beta_e + \beta_{qn}^{\pm}} \approx u_o \left(1 \mp \frac{\beta_{qn}^{\pm}}{\beta_e}\right) \quad . \tag{80}$$

But β_{qn}^{+} is slightly larger than β_q, and β_{qn}^{-} is slightly smaller than β_{qn}. Consequently, both the slow and the fast wave are moving slightly slower than they were in the solution in Section II.

The exponential form of the two phase constants, $\beta_{qn}^{\pm}$, can be reduced to the form of a correction to the polarization solution, P_z, in Section II. From Equation (49), the form of the space-charge wave harmonics is

$$\begin{matrix} \sin \\ \cos \end{matrix} \left\{ m\,\beta_{qn}\, z \right\} \quad . \tag{81a}$$

Applying the correction of Equation (79) to the above spatial variation, gives the corrected spatial variation

$$\begin{matrix}\sin\\ \cos\end{matrix}\left\{m\sqrt{1+\Delta_n^{\;2}}\;\;\beta_{qn}z\right\}(\cos\Delta_n\,\beta_{qn}z - j\sin\Delta_n\,\beta_{qn}z)\quad . \tag{81b}$$

However, Figure 12 shows that Δ_n is small compared to one and $\Delta_n^{\;2}$ would be negligible compared to one, so form (81b) reduces to the following form

$$\begin{matrix}\sin\\ \cos\end{matrix}\left\{m\beta_{qn}z\right\}(\cos\;\Delta_n\,\beta_{qn}z - j\sin\Delta_n\,\beta_{qn}z)\quad . \tag{81c}$$

As Δ_n approaches zero, form (81c) returns to form (81a), which must be the case.

The form (81c), when used to correct Equations (49), (50), and (51) yields the following results for the first two space-charge wave harmonics for polarization, velocity, and current density:

$$\begin{aligned}P_{z1} = &-\frac{J_o}{\omega_{q1}}\left(\frac{a}{2}\right)\sin\beta_{q1}z\cos\theta'\\ &- J_o\,\frac{\omega}{\omega_{q2}^{\;2}}\left(\frac{a}{4}\right)^2\left\{\left[-\sin 2\theta + \frac{3}{4\xi_{12}^{\;2}-1}\cos 2\beta_{q1}z\sin 2\theta'\right.\right.\\ &\left.+\frac{4\left(\xi_{12}^{\;2}-1\right)}{4\xi_{12}^{\;2}-1}\cos\beta_{q2}z\sin\;\theta''\right]\\ &+\left[-\frac{\omega_{q1}}{\omega}\;\frac{3}{4\xi_{12}^{\;2}-1}\sin 2\beta_{q1}z\,(\cos 2\;\Delta_1\beta_{q1}z + \cos 2\theta')\right.\\ &\left.\left.-\frac{\omega_{q2}}{\omega}\;\frac{5-8\xi_{12}^{\;2}}{4\xi_{12}^{\;2}-1}\sin\beta_{q2}z\cos\theta''\right]\right\}\quad , \end{aligned} \tag{82}$$

where

$$\theta = \omega t - \beta_e z$$

$$\theta' = \omega t - (\beta_e + \Delta_1 \beta_{q1}) z = \theta - \Delta_1 \beta_{q1} z$$

$$\theta'' = 2\omega t - (2\beta_e + \Delta_2 \beta_{q2}) z = 2\theta - \Delta_2 \beta_{q2} z \quad ;$$

$$\frac{u_z}{u_o} = 1 + \frac{a}{2} \left(\cos \beta_{q1} z \cos \theta' - \Delta_1 \sin \beta_{q1} z \sin \theta \right)$$

$$+ \left(\frac{a}{4}\right)^2 \frac{\omega}{\omega_{q2}} \left[-\frac{6\xi_{12}}{4\xi_{12}^2 - 1} \left(\sin 2\beta_{q1} z \sin 2\theta' + \Delta_1 \cos 2\beta_{q1} z \cos 2\theta' \right. \right.$$

$$\left. + \frac{\omega_{q1}}{\omega} \cos 2\beta_{q1} z \left(\cos 2\Delta_1 \beta_{q1} z + \cos 2\theta' \right) \right)$$

$$+ \left(\frac{1}{\xi_{12}} \sin 2\beta_{q1} z \sin 2\theta' + \frac{\Delta_1}{\xi_{12}} (1 + \cos 2\beta_{q1} z)(1 + \cos 2\theta') \right.$$

$$\left. + \frac{\omega_{q2}}{\omega} (1 + \cos 2\beta_{q1} z)(1 + \cos 2\theta') \right)$$

$$- \frac{4(\xi_{12}^2 - 1)}{4\xi_{12}^2 - 1} \left(\sin \beta_{q2} z \sin \theta'' + \Delta_2 \cos \beta_{q2} \cos \theta'' \right)$$

$$\left. + \frac{5 - 8\xi_{12}^2}{4\xi_{12}^2 - 1} \frac{\omega_{q2}}{\omega} \cos \beta_{q2} z \cos \theta'' \right] \quad ; \qquad (83)$$

$$
\begin{aligned}
\frac{J_z}{J_o} = & - \frac{\omega}{\omega_{q1}} \frac{a}{2} \sin \beta_{q1} z \sin \theta' \\
& + \left(\frac{\omega}{\omega_{q2}}\right)^2 \left(\frac{a}{4}\right)^2 \left[- 2 \left(\cos 2\theta + \frac{6}{4 \xi_{12}{}^2 - 1} \cos 2 \beta_{q1} z \cos 2\theta' \right.\right. \\
& \left.\left. + \frac{8(\xi_{12}{}^2 - 1)}{4 \xi_{12}{}^2 - 1} \frac{\omega_{q2}}{\omega} \cos \beta_{q2} z \cos \theta' \right) \right] \\
& + \frac{6}{4 \xi_{12}{}^2 - 1} \frac{\omega_{q1}}{\omega} \sin 2 \beta_{q1} z \sin 2\theta' \\
& - \frac{5 - 8 \xi_{12}{}^2}{4 \xi_{12}{}^2 - 1} \frac{\omega_{q2}}{\omega} \sin \beta_{q2} z \sin \theta''
\end{aligned}
\qquad (84)
$$

The Equations (82), (83), and (84) are complicated and are presented more conveniently in graphic form. The velocity for the first two space-charge wave harmonics are given in Figure 13.

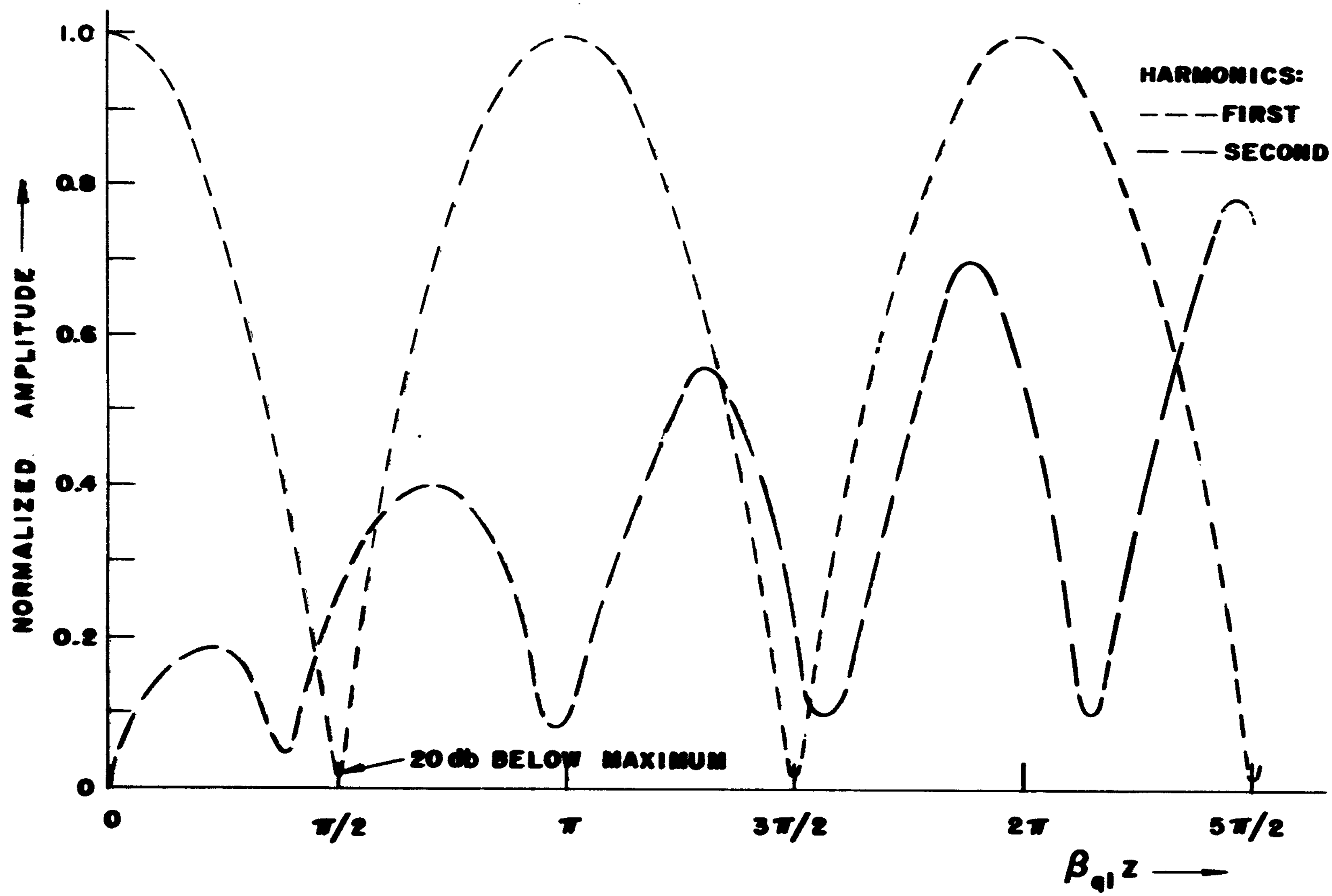

Figure 13. Corrected Velocity Harmonics versus $\beta_{ql} z$ for Finite Beam.

CONCLUSIONS AND RECOMMENDATIONS

In Section I, the electron transit time allowing for a changing electric field was described. A second order velocity correction was expanded in Appendix B, giving the final time as an infinite series of time harmonics. Since the transit time was obtained by averaging the transit times for initial and final velocity, the analysis is quite accurate for transit angles less than $\pi/2$ radians. Figure 2 clearly shows the skewing of the conventional sinusoidal transit time correction caused by the higher harmonics. This skewing represents the effects of the transcendental relation (Equation 5) between distance and time in the interaction region. Because the final velocity is an infinite Fourier series, the meaning of a gap-coupling coefficient must be changed; it can no longer be considered a constant, but a function of modulation and buncher gap width.

In Section II, the nonlinear bunching process in electron beams, including space charge effects, is solved for finite beams by the method of successive approximations. The two special cases of infinite beam and thin beam give the two extreme conditions of the general solution. For the infinite beam case, Figure 7 shows that the peaks and nulls of the current harmonics occur at the same point, and nulls are always separated by the period of $\beta_p z = \pi$. Moreover, the amplitudes for each harmonic are constant with distance. For the thin beam case, Figure 9 shows that the peaks and nulls of the nth harmonic occur in n-multiples of the fundamental period of $\beta_{q1} z = \pi$, and the amplitude rises as (n-1) power of distance. The growth of the higher harmonics physically represents the parametric transfer of energy between the fundamental and the higher harmonics.

Although the higher harmonics grow at an increasingly faster rate with distance, they saturate at a decreasingly lower amplitude. In the limit of the thin beam case, all harmonics continue to increase until they saturate.

For the finite beam case, Figure 5 shows that the period of the current harmonics decreases irrationally while the amplitudes of the peaks initially increase with distance. They are expected to fall and rise in a nearly periodic manner. The initial rise of the peaks has been experimentally verified by Mihran[8]. Both the irrational decrease in period and the harmonic growth are caused by combining the particular solution whose spatial variations are functions of the plasma phase constants of the lower harmonics, and the complementary solution whose spatial variations are functions of the plasma phase constant of the harmonic itself. The plasma phase constants are dependent on the space charge reduction factors of the n harmonics. Moreover, any one harmonic can be eliminated by varying the transverse dimensions of the beam, by varying the transverse dimensions of the beam, the a/b ratio, the ω/u_o ratio, or the location of the output cavity in order to operate the cavity at the position at the spurious harmonic current null. However, Figure 5 shows that for a fixed transverse geometry this is not the most efficient operation of the klystron, because the minimum of the higher harmonic may not occur at the maximum of the fundamental. Figure 15 shows the changing of the null position of the second and third space charge harmonics by varying the beam diameter, the a/b ratio, or the ω/u_o ratio.

Harmonics can also be eliminated because they are the sum of a forced (particular) solution and an arbitrary (complementary) solution.

By driving the klystron with a signal at the undesirable harmonic of the proper phase and amplitude, satisfying the complementary solution, this component of the harmonic can be made to cancel exactly the particular component of the harmonic caused by the lower harmonics of the beam current at the position of the output cavity.

In Section III, the interaction impedance concept illustrates that the impedance of each space-charge harmonic is actually a function of the space-charge wave propagation constant rather than the beam propagation constant, β_e. This approach yields solutions to the beam equation such that both the space charge wave phase constants for a particular ω/u_o and transverse geometry are slightly larger than those generally used. Consequently, the phase velocities are slightly smaller than those generally used. Since the fast and slow wave velocities are no longer of the same amplitude, they never cancel; that is, the velocity can only decrease to a minimum amplitude, never to zero. In higher harmonics lower order terms of the particular solution also prevent the velocity from nulling. The current, however, must go to zero and does so, but the increase in the phase constant causes a slight change in the phase and location of the null as shown in Equation (84).

APPENDIX A. TRANSIT-TIME CORRECTION FACTOR

The transit time correction factor, δ, can be expressed as accurately as needed, depending on the degree of accuracy desired.

Equation (11) can be written approximately as

$$\frac{u_f}{u_o} = 1 + \frac{\tilde{u}_f}{u_o} = 1 + 2\delta \quad , \tag{A1}$$

such that δ is a function that contains itself. From Equation (12), it is seen that

$$\delta = \frac{a}{4} \frac{\sin\left[\frac{\beta_e d}{2}\left(\frac{1}{1+\delta}\right)\right]}{\frac{\beta_e d}{2}} \cos\left[\omega t - \beta_e d + \frac{\beta_e d}{2}\left(\frac{1}{1+\delta}\right)\right] \quad . \tag{A2}$$

For an infinitesimal gap, Equation (A2) reduces to

$$\delta = \frac{a}{4}\left(\frac{1}{1+\delta}\right) \cos\omega t \quad . \tag{A3}$$

However, from Equation (12) it is seen that for an infinitesimal gap, the first approximation of δ is written as

$$\delta = \frac{a}{4} \cos\omega t \quad . \tag{A4}$$

Thus comparing Equation (A3) to Equation (A4), one sees that δ can always be replaced with $\delta/(1+\delta)$ for the next higher order correction. For example, in the limit, the correction becomes

$$\frac{u_f}{u_o} = 1 + 2 \cfrac{\delta}{1 + \cfrac{\delta}{1 + \cfrac{\delta}{1 + \delta}}} \quad . \tag{A5}$$
$$\vdots$$

A more careful observation of this series shows that

$$\frac{\delta}{1 + F(\delta)} = F(\delta) \qquad , \tag{A6}$$

where

$$F(\delta) = \cfrac{\delta}{1 + \cfrac{\delta}{1 + \delta \atop \vdots}}$$

Solving for $F(\delta)$ yields

$$F(\delta) = \frac{1}{2}\left(\sqrt{1 + 4\delta} - 1\right) \quad .$$

(Only the positive root has meaning.)

Thus combining Equations (A1), (A4), (A6), and (A7), one can write the velocity as

$$\frac{u}{u_o} = 1 + 2F(\delta) = (1 + a\cos\omega t)^{\frac{1}{2}} \quad .$$

This is exactly Webster's[1] theory, proving that δ can be corrected by always replacing it as $\frac{\delta}{1 + \delta}$ in the transit-time Equation (11) .

APPENDIX B. TRANSIT-TIME VELOCITY

The velocity of Equation (13) when expanded gives an infinite series of harmonics, and is in the form of

$$\frac{u_f}{u_o} = a_o + \sum_n a_n \cos n \left(\omega t - \frac{\beta_e d}{2}\right) \sum_n b_n \sin n \left(\omega t - \frac{\beta_e d}{2}\right) \quad . \tag{B1}$$

This series is exactly accurate for the third harmonic (within the limitations of the assumptions made). Using the following definitions:

$$a = \frac{\beta_e d}{2}$$

$$b = \frac{\beta_e d}{2} \quad \frac{\alpha\mu}{4}$$

$$c = \frac{\beta_e d}{2} \left(\frac{\alpha\mu}{4}\right)^2$$

$$\phi = \left(\omega t - \frac{\beta_e d}{2}\right) \quad ,$$

gives the complete series, through the third harmonic as follows:

$$
\begin{aligned}
\frac{\tilde{u}_{f-1}}{u_o} = \frac{a}{2}\,\frac{1}{\frac{\beta_e d}{2}} \Bigg\{ & \Big[\sin a \sin b\, J_o^2(b)\, J_o(c)\, J_1(c) - \cos a \cos b\, J_o^2(b)\, J_o(c)\, J_1(c) \Big] \\
& + \Big[\sin a \sin b\, J_o^2(b)\, J_o^2(a) - \sin a \sin b\, J_o(b)\, J_o^2(c)\, J_1(b) \\
& \quad -3 \cos a \sin b\, J_o^2(b)\, J_1^2(c) + \cos a \cos b\, J_o(b)\, J_o^2(c)\, J_1(b) \Big] \cos\phi \\
& + \Big[- \sin a \sin b\, J_o^2(b)\, J_o^2(c) + \sin a \cos b\, J_o(b)\, J_o^2(c)\, J_1(b) \\
& \quad - \cos a \cos b\, J_o^2(b)\, J_1^2(c) + \cos a \sin b\, J_o^2(b)\, J_o^2(c)\, J_1(b) \Big] \sin\phi \\
& + \Big[\sin a \sin b\, J_o^2(b)\, J_o(c)\, J_1(c) - \cos a \cos b\, J_o^2(b)\, J_o(c)\, J_1(c) \Big] \cos 2\phi \\
& + \Big[\sin a \cos b\, J_o^2(b)\, J_o(c)\, J_1(c) + \cos a \sin b\, J_o^2(b)\, J_o(a)\, J_1(c) \Big] \sin 2\phi \\
& + \Big[- \sin a \sin b\, J_o(b)\, J_o^2(c)\, J_1(b) - \cos a \sin b\, J_o^2(b)\, J_1^2(c) \\
& \qquad + \cos a \cos b\, J_o(b)\, J_o^2(c)\, J_1(b) \Big] \cos 3\phi \\
& + \Big[- \sin a \cos b\, J_o(b)\, J_o^2(c)\, J_1(b) - \cos a \cos b\, J_o^2(b)\, J_1^2(c) \\
& \qquad - \cos a \sin b\, J_o(b)\, J_o^2(c)\, J_1(b) \Big] \sin 3\phi \Bigg\} . \qquad \text{(B2)}
\end{aligned}
$$

APPENDIX C. CONSTANTS USED IN POLARIZATION EQUATIONS

The following constants are defined for use in Equations (49), (50) and (51). When one calculates these constants for a typical beam case, the ξ factors can be obtained for various a/b ratios from Figure 14. The corresponding first null points of the current harmonics are presented in Figure 15.

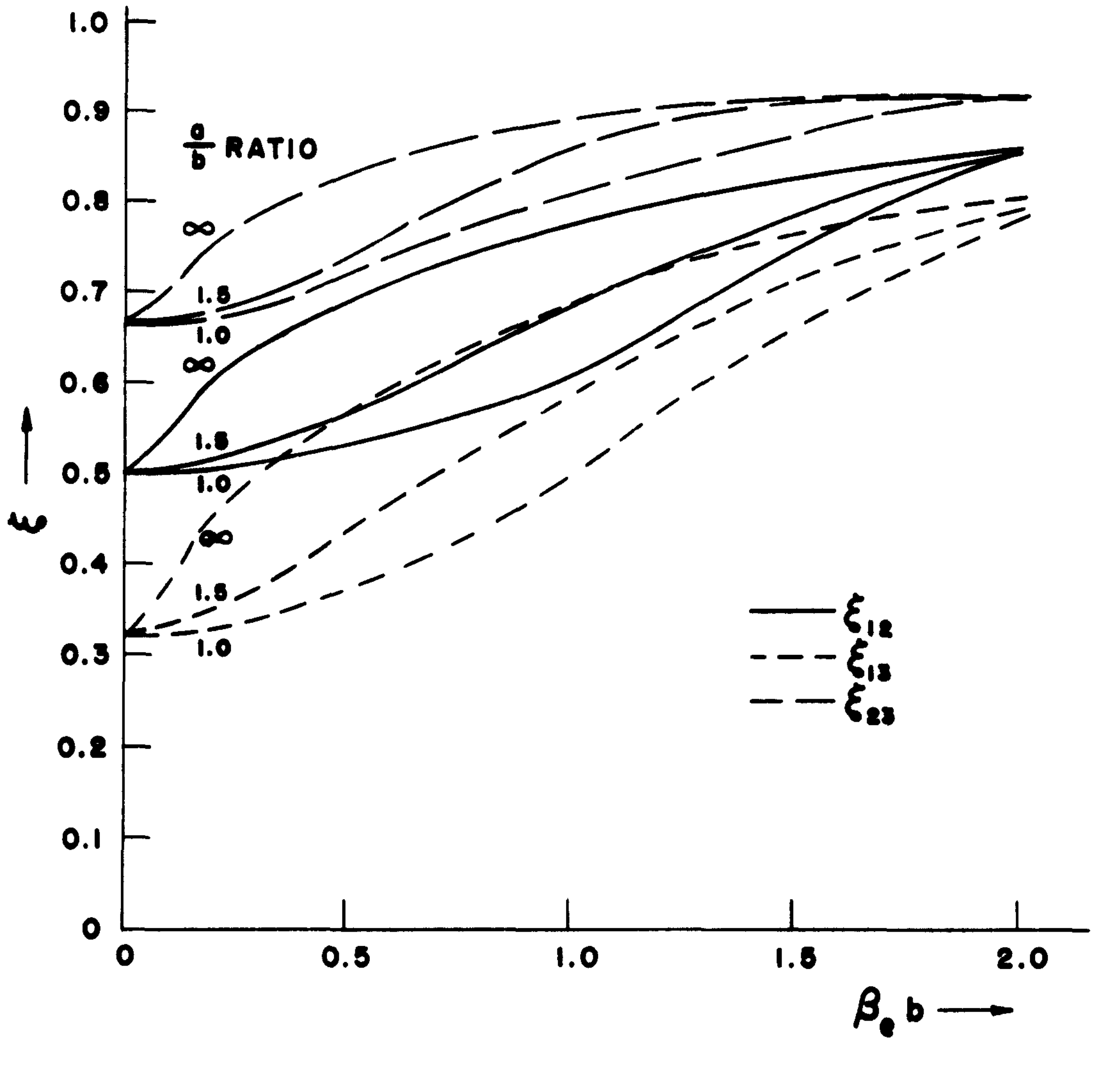

Figure 14. ξ Factors for Various a/b Ratios versus $\beta_e b$.

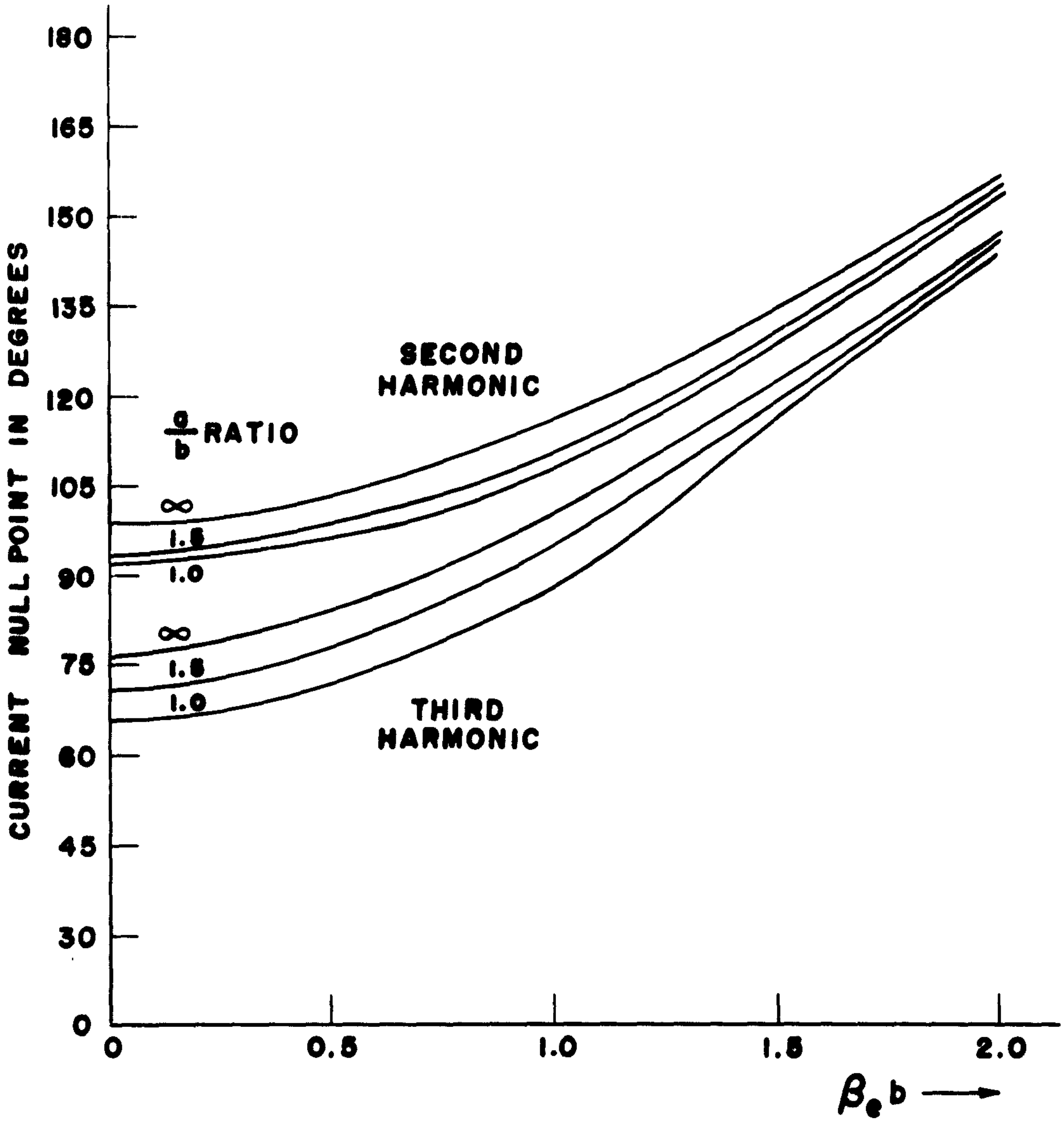

Figure 15. Current Null Point versus $\beta_e b$.

$$K_1 = \left(-\frac{3}{2}\xi_{12}{}^2 - \frac{3}{4}\right)\frac{1}{4\xi_{12}{}^2-1}\ \frac{1}{9\xi_{13}{}^2-1}$$

$$K_2 = \left[-\frac{1}{4}\xi_{12}{}^2 - \frac{1}{8} + \left(\frac{27}{8}\xi_{12}{}^2 - \frac{9}{16}\right)\frac{1}{4\xi_{12}{}^2-1}\right]\frac{1}{\xi_{13}{}^2-1}$$

$$K_3 = \frac{1}{2}\ \frac{\xi_{12}{}^2-1}{4\xi_{12}{}^2-1}\left(+\frac{1}{2} + 3\xi_{12} + \xi_{12}{}^2\right)\frac{1}{\left(\xi_{23}+\xi_{13}\right)^2-1}$$

$$K_4 = \frac{1}{2}\ \frac{\xi_{12}{}^2-1}{4\xi_{12}{}^2-1}\left(-\frac{5}{2} + 3\xi_{12} - \xi_{12}{}^2\right)\frac{1}{\left(\xi_{23}-\xi_{13}\right)^2-1}$$

$$K_5 = -\left\{3K_1 + K_2 + (\xi_{12}+1)K_3 + (\xi_{12}-1)K_4 + \frac{1}{\xi_{23}{}^2}\left[\frac{1}{4} + \left(\frac{1}{4} - \xi_{12}{}^2\right)\frac{1}{4\xi_{12}{}^2-1}\right]\right\}$$

$$K_6 = \left[-\frac{1}{4} + \left(\frac{9}{4}\xi_{12}{}^2 - \frac{3}{4}\right)\frac{1}{4\xi_{12}{}^2-1}\right]\frac{1}{9\xi_{13}{}^2-1}$$

$$K_7 = \left[-\frac{1}{4}\xi_{12}{}^2 - \frac{1}{8} + \left(-\frac{9}{8}\xi_{12}{}^2 + \frac{9}{16}\right)\frac{1}{4\xi_{12}{}^2-1}\right]\frac{1}{\xi_{13}{}^2-1}$$

$$K_8 = \frac{1}{2}\ \frac{\xi_{12}{}^2-1}{4\xi_{12}{}^2-1}\left(-\frac{1}{2} + \xi_{12} + \xi_{12}{}^2\right)\frac{1}{\left(\xi_{23}+\xi_{13}\right)^2-1}$$

$$K_9 = \frac{1}{2}\ \frac{\xi_{12}{}^2-1}{4\xi_{12}{}^2-1}\left(+\frac{5}{2} + \frac{1}{4}\xi_{12} - \xi_{12}{}^2\right)\frac{1}{\left(\xi_{23}-\xi_{13}\right)^2-1}$$

$$K_{10} = -\left[3K_6 + K_7 + (\xi_{12}+1)K_8 + (\xi_{12}-1)K_9 + \frac{1}{\xi_{23}{}^2}\left(\frac{1}{4} + \left(\frac{1}{4} - \xi_{12}{}^2\right)\frac{1}{4\xi_{12}{}^2-1}\right)\right].$$

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NONLINEAR ANALYSIS OF KLYSTRON BEAMS

J. E. Romaine

School of Electrical Engineering
CORNELL UNIVERSITY
Ithaca, New York

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NONLINEAR ANALYSIS OF KLYSTRON BEAMS

J. E. Romaine

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ABSTRACT

In the investigation of harmonic outputs from klystron amplifiers, two avenues of attack are followed. Large signal equations, valid at all drive levels and suitable for digital computer solution, are derived. Small signal equations for use in linear regions of klystrons are also derived to provide initial conditions for the large signal case. The use of the equations is illustrated in two appendices which set them up for the SAL-36, a three-cavity klystron amplifier. Since computer solutions give no indication of functional variations, analytic solutions are derived for use in synthesis. They predict the magnitudes of current density harmonics up to a drive level given by $\gamma = 0.736$, and the magnitude of the fundamental up to $\gamma = 2.8$. The main contribution to harmonic current density is seen to come from the fundamental beam disturbance. Plots of theoretical current densities as functions of γ are presented. At low values of γ, the maximum of the fundamental occurs at 90 degrees of the plasma wavelength. As γ is increased, the maximum shifts progressively closer to the plane of excitation. This has been shown experimentally by Mihran[5]. A further check of the equations is made by showing that they reduce to those of Webster in the region of negligible space charge.

INTRODUCTION

As high powered klystron amplifiers come into wider use, spurious outputs become more and more of a problem particularly in radar applications where a single output frequency is desired. The inherent nonlinearity of the bunching process causes a klystron beam to be rich in harmonics which are generally not wanted in the output. Before these undesired outputs can be minimized, a method of computing them must be developed so that proposed parameter changes can be evaluated.

Two avenues of attack present themselves. The first is to use conventional analysis to obtain solutions in closed form; the second is to use a computer to solve the defining differential equations by numerical integration. By using a computer, solutions may be obtained directly and boundary conditions fitted exactly. An immediate objection is that a computer solution gives no insight into the functional variations of the parameters involved. An analytic solution in closed form is therefore much more desirable as a tool for synthesis of klystron beams.

Obtaining an exact solution to a nonlinear equation is often extremely difficult. When the magnitudes of the variable quantities are small (small-signal case), many simplifying assumptions may be made in order to linearize the equation. Solutions then follow easily and the results obtained are valid within the range of small signals. At large signal levels the errors involved in many of the simplifying assumptions grow to such magnitudes that the results are useless. To keep errors within reasonable limits it then becomes necessary to make fewer simplifications and solve the more difficult equations.

This study employs both avenues of attack. Equations suitable for computer solution are set up for both small and large signal cases. Solutions to the small signal case are valid in the linear region of the klystron. At this point they provide initial conditions for the large signal solutions which are valid at any signal level. To aid in the synthesis problem, analytic solutions, valid up to intermediate signal levels, are set up. Although their results are questionable at very large excitations, they give the direction and order of magnitude of parameter changes necessary to minimize spurious outputs. After changing parameters, computer solutions can again be run to determine the true effects of the changes.

All equations presented may be used with any klystron. To illustrate their use, two appendices have been included to show specific equations for the SAL-36, a three-cavity klystron amplifier developed by the Sperry Gyroscope Company for the United States Air Force.

To make this paper more easily readable, the nomenclature used by others in the field will be used as much as possible. Symbols used will be defined at the beginning of each section.

SMALL SIGNAL COMPUTER SOLUTION

Symbols:

e	- electronic charge (magnitude only)
m	- electronic mass
z	- longitudinal dimension (along the beam)
ρ_o	- average space charge density (magnitude only)
$\tilde{u}_z$	- variational longitudinal electron velocity
u_o	- average electron velocity
u_z	- total longitudinal electron velocity
d	- gap spacing
V	- gap voltage
$\mu_z(z)$	- longitudinal gap-coupling coefficient
r_{ep}	- peripheral equilibrium radius of the beam
$\frac{P_z}{\rho_o}$	- variational displacement of the electrons in the longitudinal direction (from the d-c beam position)
$\frac{\ddot{P}_z}{\rho_o}$	- acceleration of the electrons in the longitudinal direction
$\tilde{E}_{cz}$	- longitudinal circuit electric field (variational)
$\tilde{E}_{sz}$	- longitudinal space charge field (variational)

Small signals are taken here to mean that the amplitude of the beam disturbance is small enough so that velocity and phase remain linear. Although harmonics are generated within the beam, they may be considered to be independent within themselves; that is, no coupling takes place between the various harmonic components. Since solutions are to be limited to linear operating regions, only the fundamental component will be considered in this analysis.

The physical conditions involved in the problem are as follows:

1. All electrons are initially moving in the positive z-direction with a uniform d-c velocity, u_o.

2. Electrons are constrained, by a strong z-directed magnetic field, to flow in a plus or minus z-direction only.

3. The d-c beam has an average electron charge density $-\rho_o$ and an equal and opposite positive ion charge density $+\rho_o$ which form a neutral plasma in which only the electrons move.

The equation of motion for electrons in the beam is given as

$$\frac{\ddot{P}_z}{\rho_o} = -\frac{e}{m} \left(\tilde{E}_{cz} + \tilde{E}_{sz}\right) \quad . \tag{1}$$

Velocity is the total time derivative of displacement and may be written

$$\tilde{u}_z = \frac{d}{dt} \frac{P_z}{\rho_o} \quad ; \tag{2}$$

but from Euler's rules of differentiation

$$\frac{d}{dt} \frac{P_z}{\rho_o} \approx \frac{\partial}{\partial t} \frac{P_z}{\rho_o} + u_o \frac{\partial}{\partial z} \frac{P_z}{\rho_o} \quad , \tag{3}$$

$$\frac{d}{dz} \frac{P_z}{\rho_o} \approx \frac{\partial}{\partial z} \frac{P_z}{\rho_o} + \frac{1}{u_o} \frac{\partial}{\partial t} \frac{P_z}{\rho_o} \quad . \tag{4}$$

Solving Equation (4) for $\frac{\partial}{\partial t} \frac{P_z}{\rho_o}$ and inserting the result in Equation (3) gives

$$\frac{\tilde{u}_z}{u_o} = \frac{d}{dz} \frac{P_z}{\rho_o} \quad . \tag{5}$$

Since acceleration is the total time derivative of velocity, it can be written as

$$\frac{d}{dt}\tilde{u}_z = \frac{\ddot{P}_z}{\rho_o} \quad . \tag{6}$$

But $\frac{d}{dt}\tilde{u}_z = u_o \frac{d}{dz}\tilde{u}_z$, therefore,

$$\frac{d}{dz}\frac{\tilde{u}_z}{u_o} = \frac{\dfrac{\ddot{P}_z}{\rho_o}}{u_o^2} \quad . \tag{7}$$

The combined results of Equations (1), (5), and (7) yield

$$\frac{d}{dz}\frac{P_z}{\rho_o} = \frac{\tilde{u}_z}{u_o} \quad , \tag{8}$$

$$\frac{d}{dz}\frac{\tilde{u}_z}{u_o} = -\frac{e}{mu_o^2}\left(\tilde{E}_{cz} + \tilde{E}_{sz}\right) \quad . \tag{9}$$

These are the main equations, which, when solved, give the velocities and polarizations necessary to define completely the state of the beam. It is now necessary to find representations for $\tilde{E}_{cz}$ and $\tilde{E}_{sz}$.

The circuit electric field for a gridded gap is a constant $\frac{V}{d}$ over the entire interaction spacing. In higher powered klystrons, where high density beams prohibit the use of grids, electric field strength becomes a function of position. It is convenient to normalize the circuit field to the gridded gap case and then to represent the field by

$$\tilde{E}_{cz} = \frac{|V|}{d}\,\mu_z(z, r)\; e^{j(\omega t + \phi)} \quad , \tag{10}$$

where $|V|$ is the magnitude of the equivalent gridded gap voltage, d is the gap spacing, and $\mu_z(z, r)$ is the longitudinal gap-coupling coefficient which gives the geometry of the electric field. Initial phase angle is accounted for by ϕ. Electric field measurements have been made in an electrolytic tank and it was found that

$$\mu_z(z, r) = A(z) + \left(\frac{r}{r_{ep}}\right)^2 B(z) \quad , \tag{11}$$

where $A(z)$ and $B(z)$ are polynomial approximations to the experimental curves. (See Appendix A.) The expression for the circuit electric field is then

$$E_{cz} = \frac{|V|}{d} \left[A(z) + \left(\frac{r}{r_{ep}}\right)^2 B(z)\right] e^{j(\omega t + \phi)} \quad , \tag{12}$$

and is to be used as the time reference at $z = 0$. Thus, in the first cavity $\phi = 0$.

Of course an excitation field in a cavity, as described by Equation (12), will excite an infinite number of space charge modes. If it is assumed that only the lowest order mode is important, then the electric field variations across the beam in the drift space can be described by a zero order Bessel function of the first kind as shown by Beck.[1] Electric field strength, in this mode, is therefore weaker at the periphery of the beam than at the center. The field in the cavity, however, varies as a constant plus a parabola and is stronger at the periphery. The higher order modes in the cavity account for this difference. A model analysis of the electric field becomes necessary if the perturbing field of the lowest order mode is to be described correctly and it can be made either before or after the equations are solved. In this small signal case the solutions will be obtained as if the field in the beam varied exactly as the field in the cavity to see what displacements and velocities would occur at the periphery if all the modes were considered. Results obtained must then be modified before they can be used as initial conditions for a large signal solution. The actual modal analysis is shown in the next section and the modification of small signal results is explained in Appendix B.

An expression for $\tilde{E}_{sz}$ is found from Poisson's equation,

$$\nabla \cdot E_s = \frac{\rho}{\epsilon_o} \quad . \tag{13}$$

Taking only the longitudinal effects into account, $\rho = \rho_o \frac{\partial}{\partial z} \frac{P_z}{\rho_o}$ and Equation (13) becomes

$$\tilde{E}_{sz} = \omega_p{}^2 \frac{m}{e} \frac{P_z}{\rho_o} \quad , \tag{14}$$

where $\frac{e\rho_o}{m\epsilon_o} = \omega_p{}^2$ (plasma frequency). Equation (14) is exactly true in the case of a beam of infinite dimensions. Since the transverse limits are finite, Equation (14) must be modified by replacing ω_p by ω_{qz}, the reduced plasma frequency. The space charge field in final form is then

$$\tilde{E}_{sz} = \omega_{qz}{}^2 \frac{m}{e} \frac{P_z}{\rho_o} \quad . \tag{15}$$

The combination of Equations (12) and (15) with Equation (9) results in

$$\frac{d}{dz} \frac{\tilde{u}_z}{u_o} = - \frac{e}{m u_o{}^2} \left\{ \frac{|V|}{d} \left[A(z) + \left(\frac{r}{r_{ep}} \right)^2 B(z) \right] e^{j(\omega t + \phi)} + \omega_{qz}{}^2 \frac{m}{e} \frac{P_z}{\rho_o} \right\} \quad . \tag{16}$$

Solutions at any value of radius are specified completely by considering the two radius values $r = 0$ and $r = r_{ep}$, since

$$\frac{P_z}{\rho_o} = \left(\frac{P_z}{\rho_o}\right)_a + \left(\frac{r}{r_{ep}}\right)^2 \left[\left(\frac{P_z}{\rho_o}\right)_p - \left(\frac{P_z}{\rho_o}\right)_a\right], \tag{17}$$

where the subscripts a and p denote axial and peripheral values respectively. Equation (16) can now be written as two separate equations:

$$\frac{d}{dz}\left(\frac{\tilde{u}_z}{u_o}\right)_a = -\frac{e}{mu_o^2}\,\frac{|V|}{d}\cdot A(z)\, e^{j(\omega t + \phi)} + \frac{\omega_{qz}^2}{u_o^2}\left(\frac{P_z}{\rho_o}\right)_a , \tag{18}$$

$$\frac{d}{dz}\left(\frac{\tilde{u}_z}{u_o}\right)_p = -\frac{e}{mu_o^2}\,\frac{|V|}{d}\left[A(z) + B(z)\right] e^{j(\omega t+\phi)} + \frac{\omega_{qz}^2}{u_o^2}\left(\frac{P_z}{\rho_o}\right)_p . \tag{19}$$

The variational polarization distances are assumed to be

$$\left(\frac{P_z}{\rho_o}\right)_a = \left[v_a(z) + j\,w_a(z)\right] e^{j(\omega t - \beta_e z)} , \tag{20}$$

$$\left(\frac{P_z}{\rho_o}\right)_p = \left[v_p(z) + j\,w_p(z)\right] e^{j(\omega t - \beta_e z)} . \tag{21}$$

When Equation (20) is substituted into Equation (8), the result is

$$\left(\frac{\tilde{u}_z}{u_o}\right)_a = \frac{d}{dz}\left[(v_a(z) + j\,w_a(z))\; e^{j(\omega t - \beta_e z)}\right] . \tag{22}$$

Equation (22) can now be combined with Equation (18) to give

$$\frac{d^2}{dz^2}\left[(v_a(z) + j\,w_a(z))\; e^{j(\omega t - \beta_e z)}\right] = -\frac{e}{mu_o^2}\;\frac{|V|}{d}\;A(z)\; e^{j(\omega t + \phi)}$$

$$+ \frac{\omega_{qz}^2}{u_o^2}\;(v_a(z) + j\,w_a(z)\; e^{j(\omega t - \beta_e z)} . \tag{23}$$

After performing the operations indicated and dividing through by $e^{j(\omega t - \beta_e z)}$, Equation (23) becomes

$$\frac{d^2}{dz^2}\;(v_a(z) + j\,w_a(z)) = -\frac{e}{mu_o^2}\;\frac{|V|}{d}\;A(z)\; e^{j(\beta_e z + \phi)} + \frac{\omega_{qz}^2}{u_o^2}\;(v_a(z) + j\,w_a(z)) . \tag{24}$$

For solution by a digital computer, Equation (24) must be separated into real and imaginary parts and the parts expanded into first order differential equations. Letting $e^{j(\beta_e z + \phi)} = \gamma + j\epsilon$, Equation (24) expands into

$$\frac{d\,v_a(z)}{dz} = v_a'(z) = p_a(z) \quad ,$$

$$\frac{d}{dz}\;p_a(z) = p_a'(z) = -\frac{e}{mu_o^2}\;\frac{|V|}{d}\;A(z)\,\gamma + \frac{\omega_{qz}^2}{u_o^2}\;v_a(z) \quad ,$$

$$\frac{d}{dz}\;w_a(z) = w_a'(z) = q_a(z) \quad , \qquad \text{--- S1} \quad r = 0$$

$$\frac{d}{dz}\, q_a(z) = q_a'(z) = -\frac{e}{mu_o^2}\, \frac{|V|}{d}\, A(z)\, \epsilon + \frac{\omega_{qz}^2}{u_o^2}\, w_a(z) \quad .$$

Similarly Equations (8), (19), and (21) can be combined to give the equations defining polarization distances and velocities at the periphery of the beam. These are

$$\frac{d}{dz}\, v_p(z) = v_p'(z) = p_p(z) \quad ,$$

$$\frac{d}{dz}\, p_p(z) = p_p'(z) = -\frac{e}{mu_o^2}\, \frac{|V|}{d}\, (A(z) + B(z))\, \gamma + \frac{\omega_{qz}^2}{u_o^2}\, v_p(z) \; ,$$

$$\frac{d}{dz}\, w_p(z) = w_p'(z) = q_p(z) \quad , \qquad \text{--- S2} \quad r = r_{ep}$$

$$\frac{d}{dz}\, q_p(z) = q_p'(z) = -\frac{e}{mu_o^2}\, \frac{|V|}{d}\, (A(z) + B(z))\, \epsilon + \frac{\omega_{qz}^2}{u_o^2}\, w_p(z) \quad .$$

Solutions to the Equations S1 and S2 will yield the values for v, w and their derivatives in z at $r = 0$ and $r = r_{ep}$. The values for v and w give the polarization distances, and multiplication of their derivatives by u_o gives the velocities. Equation (17) specifies them at all other radius values.

An IBM-650 digital computer program and a specific solution for the SAL-36 is shown in Appendix A.

LARGE SIGNAL COMPUTER SOLUTION

Symbols:

θ – relative phase of an electron beam disturbance and the reference microwave circuit disturbance $\theta = \omega t - Y$.

θ_o – initial relative phase of an electron beam disturbance and the reference microwave circuit disturbance. This is the identification parameter used to identify an electron all through the theory $\theta\big|_{Y=0} = \theta_o = \omega t\big|_{y=0}$.

$\tilde{\theta}$ – variation of phase from the initial relative phase θ_o computed in the study of the electron dynamics $\tilde{\theta} = \theta_o - \theta = \beta_e \left(\frac{P_z}{\rho_o}\right)$.

Y – normalized longitudinal component $(Y = \beta_e z)$.

$\tilde{W}(Y, \theta_o)$ – normalized variational longitudinal velocity $\tilde{W} = \frac{\tilde{u}_z}{u_o}$.

$\tilde{E}_{cz}(Y, \theta_o)$ – total longitudinal variational electric (circuit) field.

$\tilde{E}_{sz}(Y, \theta_o)$ – total longitudinal variational electric (space charge) field.

$\overleftarrow{E}_{szn}(Y)$ – nth harmonic (in θ) longitudinal electric (space charge) field.

$\overleftarrow{E}_{czn}(Y)$ – nth harmonic (in θ) longitudinal electric (circuit) field.

$E_{czn}(Y)$ – nth harmonic (in ωt) longitudinal electric (circuit) field.

$\mu_{zn}(Y, r)$ – nth harmonic (in ωt) longitudinal electric (circuit) field distribution function showing the geometrical variations for a gridless gap.

$\mu_{zn}'(Y)$ – nth harmonic (in ωt) longitudinal electric (circuit) field distribution for the lowest order space charge wave mode at $r = 0$.

V_n - nth harmonic (in ωt) gap voltage

ϕ_n - nth harmonic (in ωt) gap voltage phase angle

$\tilde{i}(Y)$ - total beam current

$\vec{i}_n(Y)$ - nth harmonic (in θ) beam current

I_n - nth harmonic (in ωt) induced gap current

Z_n - nth harmonic (in ωt) circuit impedance at the gap

$F(\theta'-\theta)$ - normalized (displacement dependent) space charge force function (force between two discs of charge located $\theta' - \theta$ apart in phase.)

$G(\theta' - \theta)$ - normalized (displacement and velocity dependent) space charge force function

δ_n - nth harmonic (in θ) longitudinal space charge reduction factor

σ_n - nth harmonic (in θ) velocity dependent longitudinal space charge reduction factor

β_e - electron phase constant or wave number $\beta_e = \frac{\omega}{u_o}$

ω_p^2 - square of the beam plasma frequency $\omega_p^2 = \frac{e\rho_o}{m\epsilon_o}$

d - longitudinal gap separation of the microwave cavity

V_o - d-c beam voltage

$-i_o$ - d-c beam current

u_o - average electron velocity

$\tilde{u}_z$ - variational longitudinal electron velocity

$\frac{P_z}{\rho_o}$ - variational displacement of the electrons in the longitudinal direction (from the d-c beam position).

As the amplitude of the beam disturbance increases, the harmonic content of the klystron rises and space charge forces become more difficult to analyze. With long interaction spaces, long drift spaces, or large applied signals, electrons may have large displacements accompanied by severe nonlinearities in phase and velocity. If a Fourier analysis is made of the various beam parameters, solutions are possible only when phase and velocity remain linear. To overcome this difficulty, partial differential equations are written for the beam parameters while physical quantities in the circuit are Fourier analyzed. The theory and methods used here are based on work done by Wang.[2]

In an electron beam, it is desirable to express the dynamical quantities in terms of an independent variable that identifies any particular electron. To do this, the entrance phase angle of the electron at a particular position is chosen as the electron identification co-ordinate. At the position chosen, the phase angle $\theta = \omega t - Y$ is defined as θ_o. At any other location the phase angle θ will not, in general, be the same as θ_o but can be related functionally as $\theta(Y, \theta_o)$. All disturbances are periodic in θ_o since it is evident that the forces experienced by a particular electron of phase θ_o will be duplicated by the electron which starts a whole number of cycles later. Because of the interrelationships among θ_o, ωt, and Y, the disturbance parameters may be expressed in terms of any two variables of the three. A complete description of the interrelationships may be given by a surface in the θ_o, ωt, Y co-ordinate system as shown in Figure 1.

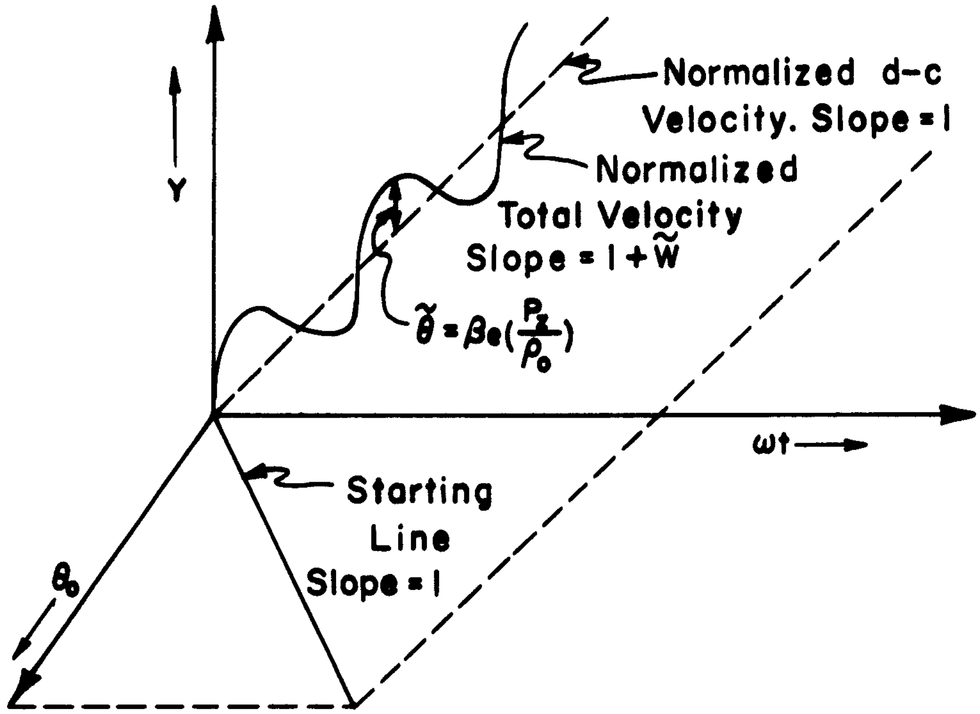

Figure 1. Interrelationships Among θ_o, ωt, and Y.

The two main working equations are derived, as in the small signal theory, from the equation of motion of electrons. From Equations (1) and (6) and Euler's rules of differentiation

$$\frac{d}{dt}\tilde{u}_z = \frac{\partial}{\partial t}\tilde{u}_z + \frac{\partial}{\partial z}\tilde{u}_z \frac{dz}{dt} \quad . \tag{25}$$

Using the relationships $\frac{dz}{dt} = u_o + \tilde{u}_z$, and $\frac{1}{2} m u_o{}^2 = eV_o$, and realizing that $\frac{d}{dt}\theta_o = 0$ since the electron identification parameter

θ_o is held constant as the electron is followed in space, Equation (25) becomes

$$(1 + \tilde{W}(Y,\theta_o)) \frac{\partial \tilde{W}(Y,\theta_o)}{\partial Y} = - \frac{(\tilde{E}_{cz} + \tilde{E}_{sz})}{2 V_o \beta_e} \quad . \tag{26}$$

From Equations (2) and (3)

$$\frac{d}{dt} \frac{P_z}{\rho_o} = \tilde{u}_z = \frac{\partial}{\partial t} \frac{P_z}{\rho_o} + \frac{dz}{dt} \frac{\partial}{\partial z} \frac{P_z}{\rho_o} \quad . \tag{27}$$

When Equation (27) is multiplied by $\frac{\beta_e}{u_o}$ it becomes

$$\frac{1}{u_o} \frac{d}{dt} \tilde{\theta}(Y,\theta_o) = \beta_e \tilde{W}(Y,\theta_o) = (1 + \tilde{W}(Y,\theta_o)) \frac{\partial}{\partial z} \tilde{\theta}(Y,\theta_o) \quad , \tag{28}$$

where $\frac{\partial}{\partial t} \tilde{\theta}(Y,\theta_o) = 0$, since $\frac{d\theta_o}{dt} = 0$ as explained previously. Realizing that $\frac{d\tilde{\theta}}{dt} = \frac{d\theta_o}{dt} - \frac{d\theta}{dt} = - \frac{d\theta}{dt}$, Equation (28) transforms into

$$\frac{\partial \tilde{\theta}(Y,\theta_o)}{\partial Y} = \frac{\tilde{W}(Y,\theta_o)}{1 + \tilde{W}(Y,\theta_o)} = - \frac{\partial \theta(Y,\theta_o)}{\partial Y} \quad . \tag{29}$$

Solutions to Equations (26) and (29) for all values of θ_o from 0 to 2π are sufficient to specify the beam completely. It remains now to determine $\tilde{E}_{cz}$ and $\tilde{E}_{sz}$ which are physical quantities that can be represented by Fourier series.

The Fourier coefficients of $\tilde{E}_{cz}(Y,\theta_o)$ are to be represented in the following manner:

$$\tilde{E}_{cz}(Y,\theta_o) = \sum_n \frac{1}{2} \left[\overleftrightarrow{E}_{czn}(Y)\, e^{jn\theta} + \overleftrightarrow{E}^*_{czn}(Y)\, e^{-jn\theta} \right] . \tag{30}$$

In essence, the Fourier coefficient of the harmonic time series is $\dot{E}_{czn} = \overleftrightarrow{E}_{czn}(Y)\, e^{jY}$, where $\overleftrightarrow{E}_{czn}(Y)$ represents an arbitrary part of the functional dependence on Y. The advantage of this representation is that, in general, the physical quantities of interest are propagating very close to the average electron beam speed and $\overline{E_{czn}}(Y)$ will be a slowly varying function of Y. Equation (30) may be written in a time series as

$$\tilde{E}_{cz}(Y, \theta_o) = \sum_n E_{czn}(Y) \cos n\omega t \quad . \tag{31}$$

It is convenient here, as in the small signal case, to normalize to an equivalent gridded gap voltage and use Equation (31) in the form

$$\tilde{E}_{cz}(Y, \theta_o) = \sum_n \frac{\mu_{zn}(Y, r) \, |V_n|}{d} \cos(n\theta + nY + \phi_n) \quad , \tag{32}$$

where ϕ_n is the phase difference between the gap voltage and the time reference of $\tilde{E}_{cz}$ in the first cavity, and $n\omega t = n\theta + nY$ by definition.

In the small signal case, the equations were set up for solutions at $r = 0$ and $r = r_{ep}$. Equation (19) could then be used to get solutions at all other radius values. As stated then, a modal analysis of the electric field must be made either before or after solutions are obtained to determine the actual fields of the important modes. In this large signal case, the analysis will be made now so that results can be used without modification.

The expansion in normal modes of the magnitude of the electric field is

$$E_{czn} = \sum_k C_{nk}(Y) \, E_{cznk} \quad , \tag{33}$$

where E_{cznk} is to be normalised so that the integral of its square over the transverse area of the beam is equal to unity. Only the lowest order space charge wave mode is being considered and for this E_{czn1} is proportional to $J_o(\beta_{tn} r)$, where β_{tn} is the nth harmonic radial propagation constant and J_o is the zeroth order Bessel function of the first kind. The summation over k in Equation (33) vanishes and E_{czn1} is found to be

$$E_{czn1} = \frac{J_o(\beta_{tn} r)}{\left[\frac{1}{\beta_{tn}^2} \int_0^{\beta_{tn} r_{ep}} 2\pi (\beta_{tn} r) J_o^2 (\beta_{tn} r) \, d(\beta_{tn} r)\right]^{\frac{1}{2}}}$$

$$= \frac{J_o(\beta_{tn} r)}{\left[\pi r_{ep}^2 (J_o^2 (\beta_{tn} r_{ep}) + J_1^2 (\beta_{tn} r_{ep}))\right]^{\frac{1}{2}}} \tag{34}$$

By multiplying both sides of Equation (33) by E_{czn1} and integrating over the transverse area, the value of $C_{nk}(Y)$ can be determined. That is

$$C_{nk}(Y) = \int_{TA} E_{czn} \, E_{czn1} \, d\,(TA) \quad . \tag{35}$$

From electrolytic tank measurements E_{czn} is known to be

$$E_{czn} = \frac{|V_n|}{d} \, \mu_{zn}(Y, r) = \frac{|V_n|}{d} \left[A(Y) + \left(\frac{r}{r_{ep}}\right)^2 B(Y)\right] . \tag{36}$$

Equation (35) can now be solved by using Equations (34) and (36)

$$C_{nk}(Y) = \frac{|V_n|}{d} \; \frac{2\pi}{\left[\pi r_{ep}^{2} \{J_o^{2}(\beta_{tn} r_{ep}) + J_1^{2}(\beta_{tn} r_{ep})\}\right]^{\frac{1}{2}}}$$

$$\left\{ \frac{A(Y)}{\beta_{tn}^{2}} \int_0^{\beta_{tn} r_{ep}} J_o(\beta_{tn} r)\, \beta_{tn}\, r \, d(\beta_{tn} r) \right.$$

$$\left. + \frac{B(Y)}{r_{ep}^{2}\, \beta_{tn}^{4}} \int_0^{\beta_{tn} r_{ep}} (\beta_{tn} r)^{3} J_o(\beta_{tn} r)\, d(\beta_{tn} r) \right\} .$$

After performing the indicated operations and using the result in Equation (33), the electric field can be represented in space by

$$E_{czn} = \frac{|V_n|}{d} \left\{ \frac{2\pi}{\left[\pi r_{ep}^{2} \{J_o^{2}(\beta_{tn} r_{ep} + J_1^{2}(\beta_{tn} r_{ep})\}\right]^{\frac{1}{2}}} \; \frac{r_{ep} J_1(\beta_{tn} r_{ep})}{\beta_{tn}} A(Y) \right.$$

$$\left. + \frac{\left[(\beta_{tn} r_{ep})^{3} J_1(\beta_{tn} r_{ep}) + 2(\beta_{tn} r_{ep})^{2} J_o(\beta_{tn} r_{ep}) - 2\beta_{tn} r_{ep} J_1(\beta_{tn} r_{ep})\right]}{r_{ep}^{2} (\beta_{tn})^{4}} B(Y) \right.$$

$$J_o(\beta_{tn} r) \quad . \qquad (37)$$

Equation (37) may be written more simply as

$$E_{czn} = \frac{|V_n|}{d} \left[K_{1n} A(Y) + K_{2n} B(Y)\right] J_o(\beta_{tn} r) \quad , \qquad (38)$$

since all other quantities are constants for a given klystron.

Equations (26) and (29) may now be solved at $r = 0$ and all other values obtained from the $J_o(\beta_{tn} r)$ variations. Comparison of Equations (32) and (38) shows that at $r = 0$,

$$\mu_{zn}(Y, 0) = \mu'_{zn}(Y) = K_{in} A(Y) + K_{2n} B(Y) \quad , \tag{39}$$

and this is to be used directly in Equation (32).

The presence of the disturbance in the electron beam gives rise to a space charge field which exerts forces on the electrons. Again the Fourier coefficients are to be represented by

$$\tilde{E}_{sz}(Y, \theta_o) = \sum_n \frac{1}{2} \left[\overrightarrow{E_{szn}}(Y)\, e^{jn\theta} + \overrightarrow{E^*_{szn}}(Y)\, e^{-jn\theta} \right] . \tag{40}$$

It is common in klystron analysis to consider the space charge field as being proportional to the Fourier component of beam current $\overrightarrow{i_n}$. Symbolically

$$\overrightarrow{E_{szn}}(Y) = j \frac{m\omega_p^2}{en\beta_e} \quad \delta_n^2 \, \frac{\overrightarrow{i_n}}{I_o} \, (Y) \tag{41}$$

where ω_p is the plasma frequency for an infinite beam and δ_n corresponds to the plasma frequency reduction factor at the nth harmonic frequency. An expression for $\overrightarrow{i_n}(Y)$ is found by operating on the Fourier series for the total beam current and applying the principle of conservation of charge. The series is

$$\tilde{i}(Y) = -i_o + \frac{1}{2} \sum_n \overrightarrow{i_n}(Y)\, e^{jn\theta} + \overrightarrow{i_n^*}(Y)\, e^{-jn\theta} \quad . \tag{42}$$

Both sides of Equation (42) are now multiplied by $e^{-jm\theta} d\theta$ and the substitution $\tilde{i}(Y)\, d\theta = -i_o d\theta_o$ is made. After integrating both sides, there remains

$$\overrightarrow{i_n}(Y) = -\frac{i_o}{\pi} \oint e^{-jn\theta} d\theta_o \quad , \tag{43}$$

which is the desired result.

If Equation (41) is now substituted into Equation (40), and Equation (43) is substituted for $\overrightarrow{i_n}(Y)$ but with θ primed to distinguish it from the θ in Equation (40), the result is

$$\tilde{E}_{sz}(Y, \theta_o) = -\frac{\omega_p^2}{\omega^2} \, 2 V_o \beta_e \oint F(\theta' - \theta) \, d\theta'_o \quad , \tag{44}$$

where

$$F(\theta'-\theta) = \frac{1}{\pi} \sum_n \frac{\delta_n^2 + \delta_n^{*2}}{2n} \sin n(\theta'-\theta) - \frac{\delta_n^2 - \delta_n^{*2}}{j2n} \cos n(\theta'-\theta) \quad .$$

It is well to stop here for a moment to consider Equation (44).

Since θ_o is periodic, a wavelength can be broken up into discs of electrons, let us say twenty. The $\theta_o = 0^\circ$ disc may then be thought of as being repeated in space every 2π radians, thus making up an entire array of discs. Other arrays may be visualized for the $\theta_o = 18^\circ$ disc and so on. If the disc being treated at the moment is a 0° disc, and it is located at θ, then the entire 18° disc array at its $\theta' + 2\pi K$ locations will apply a force to the 0° disc. This space charge force is given by $F(\theta'-\theta)$. An integration over the twenty discs will then give the total force on the 0° disc. The other nineteen discs are treated in similar fashion. Thus Equation (44) represents the usual space charge field associated with a specific disc at a specific location in space.

A more recent theory of the space charge force[2] shows that there is a component of electric field that is proportional to $\frac{d\overrightarrow{i_n}(Y)}{dY}$. Wang gives the total modified space charge field as

$$\overrightarrow{E_{sn}}(Y) = j \frac{m\omega_p^2}{en\beta_e} \delta_n^2 \frac{\overrightarrow{i_n}(Y)}{i_o} - \frac{m\omega_p^2}{en^2} \sigma_n^2 \frac{d \frac{\overrightarrow{i_n}(Y)}{i_o}}{dY} , \tag{45}$$

where σ_n^2 is another constant determined by the particular harmonic frequency and the nature of the circuit under consideration. For cylindrical drift tubes,

$$\sigma_n^2 = \beta_e \frac{\partial \delta_n^2}{\partial \beta_e} . \tag{46}$$

By direct differentiation of Equation (43),

$$\frac{d \frac{\overrightarrow{i_n}(Y)}{i_o}}{dY} = \frac{j}{\pi} \oint e^{-jn\theta} \frac{\partial n\theta}{\partial Y} d\theta_o . \tag{47}$$

Substitution of Equation (29) into Equation (47) gives

$$\frac{d \frac{\overrightarrow{i_n}(Y)}{i_o}}{dY} = - \frac{jn}{\pi} \oint e^{-jn\theta} \frac{\widetilde{W}}{1+\widetilde{W}} d\theta_o . \tag{48}$$

The second portion of Equation (45) may now be evaluated in the same fashion as the first portion after priming the functions of θ_o in Equation (48). The final expression for the space charge field is then

$$\widetilde{E}_{sz}(Y, \theta_o) = - \frac{\omega_p^2}{\omega^2} 2 V_o \beta_e \oint \left[F(\theta'-\theta) + \frac{\widetilde{W}}{1+\widetilde{W}} G(\theta'-\theta) \right] d\theta_o , \tag{49}$$

where

$$G(\theta'-\theta) = \frac{1}{\pi} \sum_n \frac{\sigma_n^2 + \sigma_n^{*2}}{2n} \sin n(\theta'-\theta) - \frac{\sigma_n^2 - \sigma_n^{*2}}{j2n} \cos n(\theta'-\theta) \quad .$$

Curves of plasma frequency reduction factor have been published by Branch and Mihran.[3]

The final quantity of interest is the induced current in a cavity. This may be computed by relating the VI product to the integral of $E_{czn} \cdot J_{zn}$ in the beam

$$V_n I_n = \int_{-\infty}^{\infty} \int_{TA} (E_{czn} \cdot J_{zn}) \, d(TA) \, dz \quad . \tag{50}$$

It must be remembered, however, that solutions are obtained at $r = 0$ and that the important parts of E_{czn} and J_{zn} vary as $J_o(\beta_{tn} r)$. An average of $J_o^2(\beta_{tn} r)$ must be taken over the transverse area of the beam and used as a factor in the integral. The average is

$$\frac{1}{\pi r_{ep}^2 \beta_{tn}^2} \int_0^{\beta_{tn} r_{ep}} 2\pi \beta_{tn} r \, J_o^2(\beta_{tn} r) \, d(\beta_{tn} r) = J_o^2(\beta_{tn} r_{ep}) + J_1^2(\beta_{tn} r_{ep}) \quad . \tag{51}$$

After dividing Equation (50) by V_n and integrating over the transverse area, the result is

$$\frac{dI_n}{dY} = \left[J_o^2(\beta_{tn} r_{ep}) + J_1^2(\beta_{tn} r_{ep}) \right] \frac{\mu'_{zn}(Y)}{\beta_e d} \overline{i_n}(Y) \, e^{-jnY} \quad , \tag{52}$$

where e^{-jnY} is included to change from a θ Fourier component to an ωt Fourier component in the cavity.

To explain how the equations presented fit together, a schematic diagram of the complete interaction phenomena is shown in Figure 2.

This simple interaction diagram explains the basic processes of interaction and the relationships among the physical quantities involved. The arrowheads indicate the direction of flow by which one physical quantity is controlled by another. Light lines indicate a simple functional relationship while heavy lines indicate differential equations connecting the two quantities.

At the top of Figure 2, in the block marked "Beam", are three quantities: $\tilde{A}(Y, \theta_o)$, the acceleration; $\tilde{W}(Y, \theta_o)$, the normalized longitudinal velocity; and $\tilde{\theta}(Y, \theta_o)$, the normalized displacement. The differential relationships I and II are generally total differential equations in time. Since the electron identification parameter is held constant, links I and II are partial differential equations with respect to the variable Y. These are given by Equations (26) and (29) respectively.

After applying a conservation of charge argument to Equation (42), link IV is established by Equation (43).

Through an equivalent circuit, the strength of the electromagnetic field can be represented by circuit currents I_n at various frequencies. Link V represents the differential equations which are the key linkages between the beam and external circuit. It is accounted for by Equation (52).

The Fourier components of the circuit field E_{czn} can be individually related to the circuit currents I_n through the complex

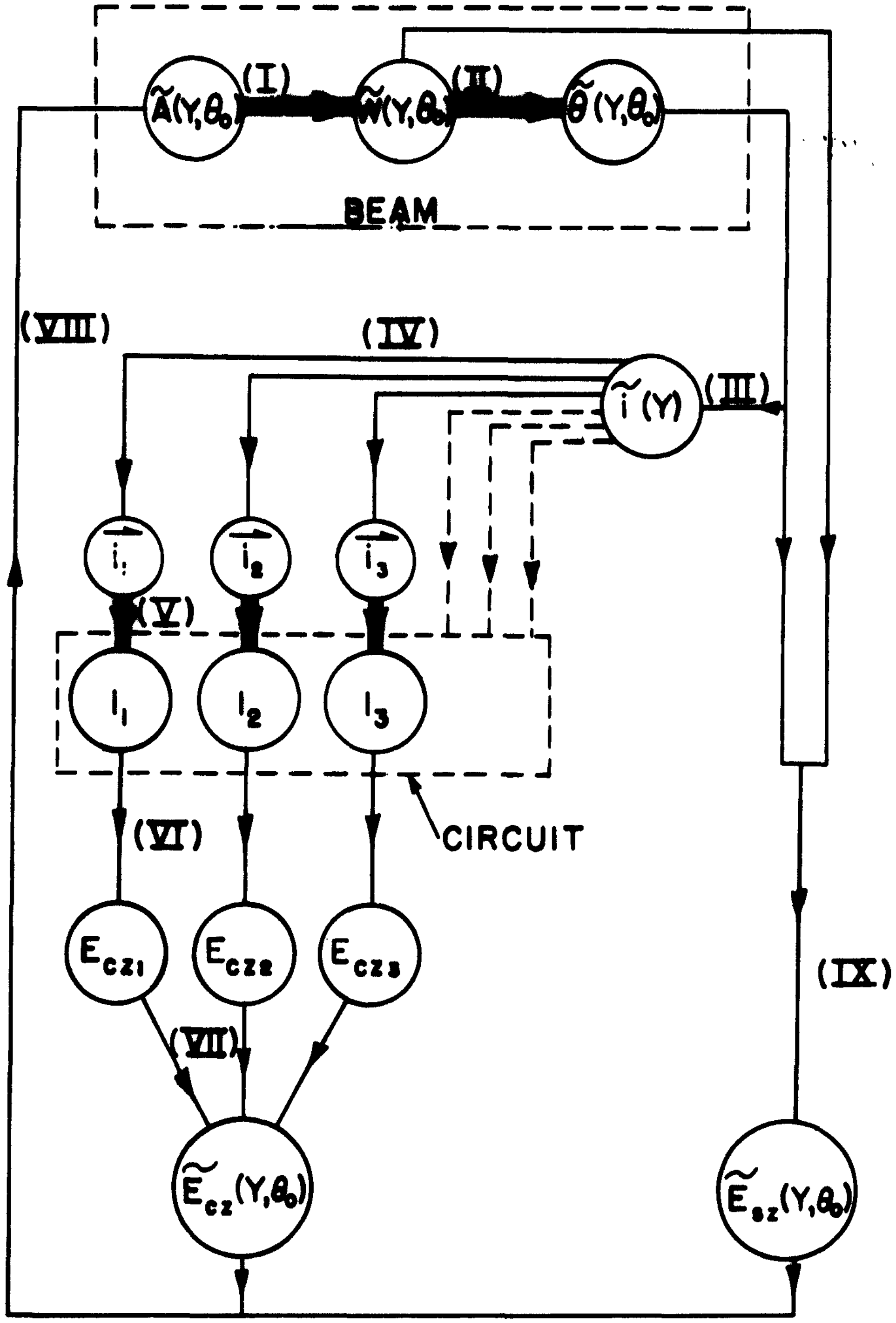

Figure 2. Interaction Diagram.

cavity impedance, Z_n. This represents link VI. In dealing with a cavity, the harmonic impedance Z_n is usually known. An initial estimate of harmonic cavity voltage V_n is made and E_{czn} determined from Equation (32) using $\mu'_{zn}(Y)$. The circuit harmonic current I_n is then solved for and an impedance Z_n' found by taking the ratio of V_n to $(-I_n)$. If $Z_n' \neq Z_n$, then a new value of V_n must be assumed and the computations run again to determine Z''_n. This process continues until the computed impedance agrees with the known impedance. Conversely, the cavity impedance required to produce a desired electric field may be computed in the same manner.

Link VII shows the combination of the Fourier components of electric field into a total field $\tilde{\tilde{E}}_{cz}(Y, \theta_o)$ and is represented by Equation (31).

In addition to the circuit force, there is another force associated with a space charge field $\tilde{\tilde{E}}_{sz}(Y, \theta_o)$, so named because it is a field related only to the beam. This force may be interpreted as a space charge repulsion force between electrons contained in the beam and is dependent upon both the velocity and displacement of the electrons. Link IX, provided by Equation (49), illustrates this dependence.

Finally, the combination of space charge force and circuit force is used to obtain the acceleration:

$$\tilde{A}(Y, \theta_o) = -\frac{e}{m} \left[\tilde{\tilde{E}}_{cz}(Y, \theta_o) + \tilde{\tilde{E}}_{sz}(Y, \theta_o)\right] \quad . \tag{53}$$

This is link VIII which completes the entire cycle of interaction.

It should be realized that large signal equations will also yield valid results under small signal conditions. The main reason for having separate small signal equations is to save computer time in the region where disturbances are relatively small.

Appendix B shows the equations set up in form suitable for numerical integration by an IBM-650 digital computer. All constants and initial conditions for the SAL-36 klystron amplifier are calculated and a workable program is given.

LARGE SIGNAL ANALYTIC SOLUTION

Symbols:

J_o - average current density

J_{zn} - nth harmonic variational longitudinal current density

P_{zn} - nth harmonic longitudinal electron polarization vector

$\mathcal{E}_z$ - total variational longitudinal electric field

m - longitudinal electric field reduction factor

u_{oo} - d-c velocity of the unmodulated electron beam

$|V|$ - magnitude of gap voltage

V_{eff} - voltage across an infinitesimal gap which has the same effect on the beam as $|V|$ in a finite gridless gap

C_1 - constant of proportionality between $|V|$ and V_{eff}

V_o - anode potential

α - depth of modulation $\left(\frac{|V|}{V_o}\right)$

ω_{qn} - nth harmonic reduced plasma frequency

β_{qn} - nth harmonic space charge wave propagation constant $\beta_{qn} = \frac{\omega_{qn}}{u_o}$

γ - amplitude parameter $\gamma = \frac{\omega}{\omega_{q1}} \frac{\alpha C_1}{2}$

S - beam cross-sectional area

Other symbols used in this section have been defined previously or are evident in the text.

A straightforward way of calculating the nonlinear behavior of klystrons is to treat the electron stream not as a limited number of electrons, but as a "fluid" where discrete charges are thought of as being "smeared out," and to solve the nonlinear space charge wave equation for given boundary conditions. The theory to be derived here always yields single velocities for given positions in space. This means that in this fluid model, electron overtaking does not occur, since if it did, the velocity would necessarily become a multi-valued function of space.

The purpose of this chapter is to derive the nonlinear space charge wave equation and solve it by third order successive approximation for a gridless gap klystron amplifier. The method of attack is based on a paper by Paschke.[4] Here again one-dimensional confined electron flow is assumed.

The analysis is begun by writing one of Maxwell's curl equations:

$$\nabla \times \underline{H} = \underline{J_c} + \frac{\partial \underline{D}}{\partial t} \quad . \tag{53}$$

When the divergence of both sides is taken this becomes

$$\nabla \cdot \nabla \times \underline{H} = \nabla \cdot \underline{J_c} + \frac{\partial}{\partial t}(\nabla \cdot \underline{D}) = 0 \quad , \tag{54}$$

since the divergence of the curl of any vector is always zero. From the assumption of confined flow $\nabla \cdot \underline{J_c} = \frac{\partial J_z}{\partial z}$. Also, from Maxwell, $\nabla \cdot \underline{D} = \rho$ and Equation (53) becomes the well known continuity equation,

$$\frac{-\partial J_z}{\partial z} = \frac{\partial \rho}{\partial t} \quad . \tag{55}$$

A new variable $\underline{P}$, the polarization, is now defined such that $\frac{-\partial P_z}{\partial t} = J_z$ and $\frac{\partial P_z}{\partial z} = \rho$ in a confined flow electron stream. It is necessary to show that $\underline{P}$ is consistent within Maxwell's equations. This is done by inserting the definition of $\underline{P}$ into Equation (53) which then becomes

$$\nabla \times \underline{H} = -\frac{\partial P_z}{\partial t}\underline{k} + \frac{\partial \underline{D}}{\partial t} = \frac{\partial}{\partial t}(\underline{D} - P_z\underline{k}) = \frac{\partial}{\partial t}(\epsilon_o\underline{E} - P_z\underline{k}) \quad . \tag{56}$$

Again taking the divergence,

$$\nabla \cdot \nabla \times \underline{H} = 0 = \frac{\partial}{\partial t}\left(\nabla \cdot \underline{D} - \frac{\partial P_z}{\partial z}\right) = \frac{\partial}{\partial t}(\rho - \rho) = 0 \quad , \tag{57}$$

which establishes the validity of the variable.

An examination of Equation (56) shows that the variable $\underline{P}$ is actually the negative of conventional polarization. This choice was made because the conventional polarization vector points in the direction of motion of positive charge and electron motion is of primary interest in klystron analysis.

The equation of motion for confined flow electrons in an axial magnetic field is given by

$$\frac{d}{dt}\tilde{u}_z = -\frac{e}{m}\tilde{E}_z = \eta\tilde{E}_z \quad . \tag{58}$$

Since the perturbation velocity $\tilde{u}_z$ is a function of both space and time, Equation (58) can be written as

$$\frac{\partial \tilde{u}_z}{\partial t} + (u_o + \tilde{u}_z)\frac{\partial \tilde{u}_z}{\partial z} = \eta\tilde{E}_z \quad . \tag{59}$$

Expressions for $\tilde{E}_z$ and $\tilde{u}_z$ must now be obtained in terms of P_z. This is done by making use of Maxwell's divergence equation $\nabla \cdot \underline{E} = \frac{\rho}{\epsilon_o}$. After taking partials and rearranging ,

$$\epsilon_o\frac{\partial \tilde{E}_z}{\partial z} + \epsilon_o\left(\frac{\partial \tilde{E}_z}{\partial x} + \frac{\partial \tilde{E}_z}{\partial y}\right) = \rho = \frac{\partial P_z}{\partial z} \quad . \tag{60}$$

When the transverse geometry is finite, the transverse electric field components exist and cause a reduction of the longitudinal electric field for a given polarization. The parameter m is introduced to account for this reduction:

$$P_z = \epsilon_o \tilde{E}_z + m \quad . \tag{61}$$

Taking the partial with respect to z and again with respect to t results in

$$\frac{\partial}{\partial z} \frac{\partial P_z}{\partial t} = \frac{\partial}{\partial t} \epsilon_o \left(\frac{\partial E_z}{\partial z} + \frac{\partial m}{\partial z} \right) \quad . \tag{62}$$

From Equations (60) and (61) and the definition of P_z, Equation (62) reduces to $-\frac{\partial}{\partial z} J_z = \frac{\partial}{\partial t} \rho$, which is the continuity equation. This establishes the consistency of the definition of m. Solving Equation (61) for $\tilde{E}_z$ gives

$$\tilde{E}_z = \frac{P_z - m}{\epsilon_o} \quad . \tag{63}$$

The convection current density in the beam is the product of charge density and velocity. Symbolically,

$$J_o + J_z = (\rho_o + \rho)(u_o + \tilde{u}_z) \quad , \tag{64}$$

where the d-c current density $J_o = \rho_o u_o$. Solving for $\tilde{u}_z$ gives

$$\tilde{u}_z = \frac{J_z - u_o \rho}{\rho_o + \rho} \quad . \tag{65}$$

In terms of P_z, Equation (65) is

$$\tilde{u}_z = \frac{\frac{\partial P_z}{-\partial t} + u_o \frac{\partial P_z}{\partial z}}{\rho_o + \frac{\partial P_z}{\partial z}} \quad . \tag{66}$$

The derived expressions for $\tilde{E}_z$ and $\tilde{u}_z$ are now inserted into the equation of motion, which, after performance of the indicated operations, transforms to

$$\left(\frac{\partial^2 P_z}{\partial t^2} + u_o \frac{\partial^2 P_z}{\partial z \partial t}\right)\left(\rho_o + \frac{\partial P_z}{\partial z}\right) - \frac{\partial^2 P_z}{\partial z \partial t}\left(\frac{\partial P_z}{\partial t} + u_o \frac{\partial P_z}{\partial z}\right)\left(\rho_o + \frac{\partial P_z}{\partial z}\right)$$

$$+ \left(J_o - \frac{\partial P_z}{\partial t}\right)\left(\frac{\partial^2 P_z}{\partial z \partial t} + u_o \frac{\partial^2 P_z}{\partial z^2}\right)\left(\rho_o + \frac{\partial P_z}{\partial z}\right) - \left(J_o - \frac{\partial P_z}{\partial t}\right)\frac{\partial^2 P_z}{\partial z^2}$$

$$\left(\frac{\partial P_z}{\partial t} + u_o \frac{\partial P_z}{\partial z}\right) + \frac{\eta}{\epsilon_o}(P_z - m)\left(\rho_o + \frac{\partial P_z}{\partial z}\right)^3 = 0 \quad . \qquad (67)$$

Equation (67) is a nonlinear differential equation which, when solved, yields expressions for polarization and, indirectly, velocity as functions of time and space. As previously stated, a knowledge of the polarization and velocity defines completely the state of the beam. A third order theory will be considered and hence terms of higher than third order are to be neglected. Under these conditions Equation (67) becomes

$$\frac{\partial^2 P_z}{\partial z^2} + \frac{2}{u_o}\frac{\partial^2 P_z}{\partial z \partial t} + \frac{1}{u_o^2}\frac{\partial^2 P_z}{\partial t^2} + \beta_p^2 (P_z - m) + \frac{2}{J_o u_o}$$

$$\left(\frac{\partial P_z}{\partial z}\frac{\partial^2 P_z}{\partial t^2} - \frac{\partial P_z}{\partial t}\frac{\partial^2 P_z}{\partial z \partial t}\right) + \frac{2}{J_o}\left(\frac{\partial P_z}{\partial z}\frac{\partial^2 P_z}{\partial z \partial t} - \frac{\partial P_z}{\partial t}\frac{\partial^2 P_z}{\partial z^2}\right)$$

$$+ \frac{3\eta}{\epsilon_o u_o^2}\frac{\partial P_z}{\partial z}(P_z - m) + \frac{1}{J_o^2}\left[\frac{\partial^2 P_z}{\partial z^2}\left(\frac{\partial P_z}{\partial t}\right)^2 - 2\frac{\partial^2 P_z}{\partial z \partial t}\frac{\partial P_z}{\partial z}\frac{\partial P_z}{\partial t}\right.$$

$$\left. + \frac{\partial^2 P_z}{\partial t^2}\left(\frac{\partial P_z}{\partial z}\right)^2\right] + \frac{3\eta}{\epsilon_o J_o u_o}\left(\frac{\partial P_z}{\partial z}\right)^2 (P_z - m) = 0 \quad . \qquad (68)$$

For very low power levels, all but first order terms can be neglected. Equation (68) would then consist of only the first four terms and would lead to the well-known space-charge wave equation for small signals. A third order solution however, requires that

$$P_z = P_{z1} + P_{z2} + P_{z3} \quad ,$$

$$\tilde{u}_z = \tilde{u}_{z1} + \tilde{u}_{z2} + \tilde{u}_{z3} \quad ,$$

$$m = m_1 + m_2 + m_3 \quad . \tag{69}$$

Substituting Equation (69) into (68) and separating terms into first, second, and third order, results in

$$f_1 + f_2 + f_3 = 0 \quad , \tag{70}$$

where

$$f_1 = \frac{\partial^2 P_{z1}}{\partial z^2} + \frac{2}{u_o} \frac{\partial^2 P_{z1}}{\partial z\, \partial t} + \frac{1}{u_o^2} \frac{\partial^2 P_{z1}}{\partial t^2} + \beta_{q1}^{\ 2} P_{z1} \quad ,$$

$$\begin{aligned} f_2 = {} & \frac{\partial^2 P_{z2}}{\partial z^2} + \frac{2}{u_o} \frac{\partial^2 P_{z2}}{\partial z\, \partial t} + \frac{1}{u_o^2} \frac{\partial^2 P_{z2}}{\partial t^2} + \beta_{q2}^{\ 2} P_{z2} \\ & + \frac{2}{J_o u_o} \left(\frac{\partial P_{z1}}{\partial z} \frac{\partial^2 P_{z1}}{\partial t^2} - \frac{\partial P_{z1}}{\partial t} \frac{\partial^2 P_{z1}}{\partial z\, \partial t} \right) \\ & + \frac{2}{J_o} \left(\frac{\partial P_{z1}}{\partial z} \frac{\partial^2 P_{z1}}{\partial z\, \partial t} - \frac{\partial P_{z1}}{\partial t} \frac{\partial^2 P_{z1}}{\partial z^2} \right) + \frac{3\eta}{\epsilon_o u_o^2} \frac{\partial P_{z1}}{\partial z} P_{z1} \quad , \end{aligned}$$

$$f_3 = \frac{\partial^2 P_{z3}}{\partial z^2} + \frac{2}{u_o} \frac{\partial^2 P_{z3}}{\partial z\, \partial t} + \frac{1}{u_o^2} \frac{\partial^2 P_{z3}}{\partial t^2} + \beta_{q3}^{\ 2} P_{z3}$$

$$+ \frac{2}{J_o u_o} \left(\frac{\partial P_{z1}}{\partial z} \frac{\partial^2 P_{z2}}{\partial t^2} + \frac{\partial P_{z2}}{\partial z} \frac{\partial^2 P_{z1}}{\partial t^2} - \frac{\partial P_{z2}}{\partial t} \frac{\partial^2 P_{z1}}{\partial z\, \partial t} - \frac{\partial P_{z1}}{\partial t} \frac{\partial^2 P_{z2}}{\partial z\, \partial t} \right)$$

$$+ \frac{2}{J_o} \left(\frac{\partial P_{z1}}{\partial z} \frac{\partial^2 P_{z2}}{\partial z\, \partial t} + \frac{\partial P_{z2}}{\partial z} \frac{\partial^2 P_{z1}}{\partial z\, \partial t} - \frac{\partial P_{z1}}{\partial t} \frac{\partial^2 P_{z2}}{\partial z^2} - \frac{\partial P_{z2}}{\partial t} \frac{\partial^2 P_{z1}}{\partial z^2} \right)$$

$$+ \frac{1}{J_o^2} \left(\frac{\partial^2 P_{z1}}{\partial z^2} \left(\frac{\partial P_{z1}}{\partial t} \right)^2 - 2 \frac{\partial P_{z1}}{\partial t} \frac{\partial P_{z1}}{\partial z} \frac{\partial^2 P_{z1}}{\partial z\, \partial t} + \frac{\partial^2 P_{z1}}{\partial t^2} \left(\frac{\partial P_{z1}}{\partial z} \right)^2 \right)$$

$$+ \frac{3\eta}{{}_o u_o^2} \left(\frac{\partial P_{z2}}{\partial z} P_{z1} + P_{z2} \frac{\partial P_{z1}}{\partial z} \right) + \frac{3\eta}{\epsilon_o u_o J_o} P_{z1} \left(\frac{\partial P_{z1}}{\partial z} \right)^2 \quad .$$

The effects of the transverse fields, represented by m, result in a reduction of the plasma frequency and are accounted for in Equation (70) by the substitution of β_{qn} for β_{pn}.

There are many possible solutions to Equation (70), the simplest of which is to set the terms of equal order equal to zero. That is,

$$f_1 = 0 \quad , \tag{71a}$$

$$f_2 = 0 \quad , \tag{71b}$$

$$f_3 = 0 \quad . \tag{71c}$$

Although this may appear to be a trivial solution, it is sufficient to satisfy the boundary conditions.

Equations (71) now represent a system of <u>linear</u> <u>differential</u> <u>equations</u>, all terms in each equation being of the same order. The necessary procedure is first to solve Equation (71a); put the solution into Equation (71b), solve it, put the solutions to Equation (71a) and (71b) into Equation (71c) and solve it. The total solutions $P_{z1} + P_{z2} + P_{z3}$ must satisfy the boundary conditions. This approach to the solution of a nonlinear differential equation is known as the method of successive approximations.

From Equations (66) and (69), the velocity terms are found to be

$$-\rho_o \tilde{u}_{z1} = \frac{\partial P_{z1}}{\partial t} + u_o \frac{\partial P_{z1}}{\partial z} \quad , \tag{72a}$$

$$-\rho_o \tilde{u}_{z2} = \frac{\partial P_{z2}}{\partial t} + u_o \frac{\partial P_{z2}}{\partial z} + \tilde{u}_{z1} \frac{\partial P_{z1}}{\partial z} \quad , \tag{72b}$$

$$-\rho_o \tilde{u}_{z3} = \frac{\partial P_{z3}}{\partial t} + u_o \frac{\partial P_{z3}}{\partial z} + \tilde{u}_{z2} \frac{\partial P_{z1}}{\partial z} + \tilde{u}_{z1} \frac{\partial P_{z2}}{\partial z} \ . \tag{72c}$$

Velocity Modulation by a Sinusoidal RF Voltage:

A gridless gap klystron cavity of width d, and center at $z = 0$ is assumed to be modulating a beam of d-c velocity u_{oo}. The effective r-f voltage across the gap is determined from

$$V_{eff} = \int_{-\infty}^{\infty} \tilde{E}_{cz} \, dz \quad . \tag{73}$$

After expanding in normal modes, the fundamental component of electric field for the first order space charge wave mode may be written as

$$\tilde{E}_{cz} = \frac{|V|}{d}\left[K_1 A(z) + K_2 B(z)\right] J_o(\beta_{t1} r)\cos(\omega t + \phi) \quad , \tag{74}$$

where all terms are defined in the previous section and solutions are to be found at $r = 0$. The limits of integration in Equation (73) may be taken as $-1.65\,d$ to $+1.65\,d$ since the amplitude of the field has fallen essentially to zero at these limits. After substitution of Equation (74), the effective voltage is

$$V_{eff} = \int_{-1.65\,d}^{1.65d} \frac{|V|}{d}\left[K_1 A(z) + K_2 B(z)\right]\cos(\beta_e z + \omega t_o + \phi)\,dz \quad , \tag{75}$$

where t_o is the time an arbitrary electron passes through the center of the cavity and $\omega t \cong \beta_e z + \omega t_o$ in the limits of the cavity. The integration is a simple but lengthy process and the result is

$$V_{eff} = C_1\,|V|\cos(\omega t_o + \phi) \quad . \tag{76}$$

The constant C_1 is given by the somewhat formidable expression

$$\begin{aligned} C_1 = {} & \frac{2}{\beta_e d}\sin\tau/2\,(K_1 a_o + K_2 b_o) \\ & + \frac{2}{\beta_e{}^3 d}(D_5 \sin\tau/2 + D_6\cos\tau/2)(K_1 a_2 + K_2 b_2) \\ & + \frac{2}{\beta_e{}^5 d}(D_3 \sin\tau/2 + D_4\cos\tau/2)(K_1 a_4 + K_2 b_4) \\ & + \frac{2}{\beta_e{}^7 d}(D_1 \sin\tau/2 + D_2\cos\tau/2)(K_1 a_6 + K_2 b_6) \quad , \end{aligned} \tag{77}$$

where $\tau/2 = 1.65\,\beta_e d$, a_K and b_K are the coefficients of $A(z)$ and $B(z)$ respectively, and

$$D_1 = (\tau/2)^6 - 30(\tau/2)^4 + 360(\tau/2)^2 - 720 \quad ,$$

$$D_2 = 6(\tau/2)^5 - 120(\tau/2)^3 + 720(\tau/2) \quad ,$$

$$D_3 = (\tau/2)^4 - 12(\tau/2)^2 + 24 \quad ,$$

$$D_4 = 4(\tau/2)^3 - 24(\tau/2) \quad ,$$

$$D_5 = (\tau/2)^2 - 2 \quad ,$$

$$D_6 = 2(\tau/2) \quad .$$

The effect of the constant C_1 is to reduce the problem to that of an infinitesimal gap located at $z = 0$ and modulated by an effective voltage of magnitude $C_1\,|V|$.

From conservation of energy,

$$\left.\frac{dz}{dt}\right|_{z_o} = u_{oo}\,(1 + a\,C_1\,\cos\,(\omega t_o + \phi))^{1/2} \quad , \tag{78}$$

where a is the depth of modulation $\frac{|V|}{V_o}$. For normal klystron operation $|a| < 1$ and Equation (78) can be developed into a series. Omitting terms of higher than third order results in

$$\left.\frac{dz}{dt}\right|_{z=0} = u_{oo}\left[1 - \frac{(aC_1)^2}{16} + \frac{aC_1}{2}\left(1 + \frac{3(aC_1)^2}{32}\right)\cos\,(\omega t_o + \phi)\right.$$
$$\left. - \frac{(aC_1)^2}{16}\,\cos\,2(\omega t_o + \phi) + \frac{(aC_1)^3}{64}\,\cos\,3(\omega t_o + \phi)\right] . \tag{79}$$

Equation (79) and the fact that at $z = 0$; $P_z = 0$, are the required initial conditions for solution of Equation (71). Equation (71a) is solved by letting $P_{z1} = P_{z1}(z)\, e^{j\omega t} + P^*_{z1}(z)\, e^{-j\omega t}$ and separating it into the two equations

$$\frac{d^2P_{z1}(z)}{dz^2} + j\,2\beta_e \frac{dP_{z1}(z)}{dz} + (\beta_{q1}{}^2 - \beta_e{}^2)\, P_{z1}(z) = 0 \quad ,$$

$$\frac{d^2P^*_{z1}(z)}{dz^2} - j\,2\beta_e \frac{dP^*_{z1}(z)}{dz} + (\beta_{q1}{}^2 - \beta_e{}^2)\, P^*_{z1}(z) = 0 \quad . \qquad (80)$$

These two equations are of the form

$$\frac{d^2y(x)}{dx^2} + \gamma_1 \frac{dy(x)}{dx} + \gamma_2\, y(x) = 0 \quad , \qquad (81)$$

which has a solution of the form $y(x) = a_1\, e^{\alpha_1 x} + b_1\, e^{\alpha_2 x}$, where $\alpha_1 = \dfrac{-\gamma_1 + \sqrt{\gamma_1{}^2 - 4\gamma_2}}{2}$, and $\alpha_2 = \dfrac{-\gamma_1 - \sqrt{\gamma_1{}^2 - 4\gamma_2}}{2}$. After applying Equation (81) to (80) and using initial conditions to determine the arbitrary constants, the solution to Equation (71a) is

$$P_{z1} = \frac{-J_o \dfrac{\alpha C_1}{2}\left(1 + \dfrac{3(\alpha C_1)^2}{32}\right)}{\omega_{q1}} \sin \beta_{q1} z \, \cos(\omega t - \beta_e z) \quad . \qquad (82)$$

Having solved Equation (71a), the result is used in Equation (71b) which is of the same form as Equation (71a) except for the driving terms provided by Equation (71a). Since the equations are linear, superposition can be used to handle each driving term separately. The arbitrary constants are again determined from the initial conditions. Equation (71c) is then treated in the same straightforward manner. The sum of the solutions to Equations (71) yield the desired result, which , after a great deal of algebraic manipulation is

$$
\begin{aligned}
P_z = &- \frac{J_o}{\omega_{q1}} \frac{aC_1}{2} \sin\beta_{q1} z \cos(\omega t - \beta_e z) + \frac{J_o}{4}\left(\frac{aC_1}{2}\right)^2 \left[\frac{\omega}{\omega_{q1}^{\,2}} (1 - \cos 2\beta_{q1} z) \cos 2(\omega t - \beta_e z)\right. \\
&\left. + \frac{1}{\omega_{q2}} \sin\beta_{q2} z \cos 2(\omega t - \beta_e z) - \frac{1}{\omega_{q1}} \sin 2\beta_{q1} z\, (1 - \cos 2(\omega t - \beta_e z)\right] \\
&+ \frac{J_o}{16}\left(\frac{aC_1}{2}\right)^3 \left[\frac{\omega^2}{2\omega_{q1}^{\,3}} (3 \sin\beta_{q1} z - \sin 3\beta_{q1} z) \cos(\omega t - \beta_e z) - \frac{7}{\omega_{q1}} \sin\beta_{q1} z \cos(\omega t - \beta_e z)\right. \\
&+ \frac{9}{\omega_{q1}} (1 - \cos 2\beta_{q1} z) \sin\beta_{q1} z \cos(\omega t - \beta_e z) + \frac{4}{\omega_{q2}} \sin\beta_{q1} z \cos\beta_{q2} z \cos(\omega t - \beta_e z) \\
&\left. - \frac{2\omega}{\omega_{q1}\omega_{q2}} \sin\beta_{q1} z \sin\beta_{q2} z \sin(\omega t - \beta_e z) + \frac{6\omega}{\omega_{q1}\omega_{q2}} (1 - \cos 2\beta_{q1} z) \cos\beta_{q2} z \sin(\omega t - \beta_e z)\right] \\
&- \frac{3J_o}{8}\left(\frac{aC_1}{2}\right)^3 \left[\frac{\frac{\omega^2}{\omega_{q1}^{\,3}} + \frac{1}{\omega_{q1}}}{4} (3 \sin\beta_{q1} z - \sin 3\beta_{q1} z) \cos 3(\omega t - \beta_e z)\right. \\
&+ \frac{\omega}{\omega_{q1}\omega_{q2}} (1 - \cos 2\beta_{q1} z) \cos\beta_{q2} z \sin 3(\omega t - \beta_e z) + \frac{\omega}{\omega_{q1}\omega_{q2}} \sin\beta_{q1} z \sin\beta_{q2} z \sin 3(\omega t - \beta_e z) \\
&\left. + \frac{2}{3\omega_{q1}\omega_{q2}} \sin\beta_{q1} z \cos\beta_{q2} z \cos 3(\omega t - \beta_e z) - \frac{1}{\omega_{q3}} \sin\beta_{q3} z \cos 3(\omega t - \beta_e z)\right] . \qquad (83)
\end{aligned}
$$

If velocity is desired it can now be computed using Equations (72).

Many of the higher order terms in Equation (83) can be neglected as long as $\frac{\omega_{qn}}{\omega} < 1$ and $\frac{aC_1}{2} < 1$. Under these conditions Equation (83) reduces to

$$
\begin{aligned}
P_z = & - \frac{J_o}{\omega_{q1}} \frac{aC_1}{2} \sin\beta_{q1}(z) \cos(\omega t - \beta_e z) \\
& + \frac{J_o}{4} \left(\frac{aC_1}{2}\right)^2 \frac{\omega}{\omega_{q1}^2} (1 - \cos 2\beta_{q1} z) \cos 2(\omega t - \beta_e z) \\
& + \frac{J_o}{32} \left(\frac{aC_1}{2}\right)^3 \frac{\omega^2}{\omega_{q1}^3} (3 \sin\beta_{q1} z - \sin 3\beta_{q1} z) \cos(\omega t - \beta_e z) \\
& - \frac{3J_o}{32} \left(\frac{aC_1}{2}\right)^3 \frac{\omega^2}{\omega_{q1}^3} (3 \sin\beta_{q1} z - \sin 3\beta_{q1} z) \cos 3(\omega t - \beta_e z) \quad .
\end{aligned}
\tag{84}
$$

Current density is simply the negative partial time derivative of P_z. From Equation (84),

$$
\begin{aligned}
\frac{J_z}{J_o} = & - \frac{\omega}{\omega_{q1}} \frac{aC_1}{2} \sin\beta_{q1} z \sin(\omega t - \beta_e z) \\
& + \frac{1}{2} \left(\frac{\omega}{\omega_{q1}}\right)^2 \left(\frac{aC_1}{2}\right)^2 (1 - \cos 2\beta_{q1} z) \sin 2(\omega t - \beta_e z) \\
& + \frac{1}{32} \left(\frac{\omega}{\omega_{q1}}\right)^3 \left(\frac{aC_1}{2}\right)^3 (3 \sin\beta_{q1} z - \sin 3\beta_{q1} z) \sin(\omega t - \beta_e z) \\
& - \frac{9}{32} \left(\frac{\omega}{\omega_{q1}}\right)^3 \left(\frac{aC_1}{2}\right)^3 (3 \sin\beta_{q1} z - \sin 3\beta_{q1} z) \sin 3(\omega t - \beta_e z) \quad .
\end{aligned}
\tag{85}
$$

Equation (85) is a somewhat surprising result since it indicates that the main contribution to harmonic current density comes from the fundamental beam disturbance and the harmonic spatial distribution is controlled by the fundamental reduced plasma phase constant.

The question arises as to how large the amplitudes of the harmonics can get and still represent good approximations to the exact solutions. To extrapolate to higher harmonics, saturation effects are neglected and the "quasi-linear" case is considered. The maximum current density appears at $\beta_{q1} z = \frac{\pi}{2}$ and for quasi-linearity is given by

$$\left.\frac{J_z}{J_o}\right|_{\beta_{q1} z = \frac{\pi}{2}} = -\gamma \sin(\omega t - \frac{\pi\beta_e}{2\beta_{q1}}) + \gamma^2 \sin 2(\omega t - \frac{\pi\beta_e}{2\beta_{q1}}) - \frac{9}{8}\gamma^3 \sin 3(\omega t - \frac{\pi\beta_e}{2\beta_{q1}}) , \tag{86}$$

where $\gamma = \frac{\omega}{\omega_{q1}} \frac{\alpha C_1}{2}$. This equation appears to be the first part of the infinite series

$$\left.\frac{J_z}{J_o}\right|_{\beta_{q1} z = \frac{\pi}{2}} = \sum 2 \frac{\left(\frac{-n}{2}\right)^n}{n!} \gamma^n \sin n(\omega t - \frac{\pi\beta_e}{2\beta_{q1}}) . \tag{87}$$

Using the ratio test and realizing that $x = e^{\ln x}$, the series is found to be absolutely convergent for $\gamma < \frac{2}{e} \cong 0.736$. Equation (85) appears to be a valid approximation for <u>total</u> current below the level of $\gamma = 0.736$. However, this does not necessarily mean that the <u>fundamental</u> current at $\beta_{q1} z = \frac{\pi}{2}$ is not correct beyond this level. It is the limitation to third and lower order terms which prevents the prediction of the saturation behavior of the harmonics. The power series for the fundamental, from Equation (85), is

$$\left.\frac{J_{z1}}{J_o}\right|_{\beta_{q1} z = \frac{\pi}{2}} = -\gamma\left(1 - \frac{1}{8}\gamma^2 + \ldots\right) \sin\left(\omega t - \frac{\pi\beta_e}{2\beta_{q1} z}\right) \quad . \tag{88}$$

This series converges much more rapidly than the one in Equation (87) and at the saturation level, easily shown to be $\gamma = \sqrt{\frac{8}{3}}$, the second term is only one-third of the first term. At any other value of $\beta_{q1} z$, saturation will be reached at a different γ value and, if desired, the values of γ required for saturation at any $\beta_{q1} z$ can be computed.

Figure 3 shows plots of current density versus distance for the fundamental, second harmonic, and third harmonic at $\gamma = 0.73$, the upper limit of validity of exact harmonic determination. In Figure 4, the fundamental current density is plotted for various values of γ. The shift in the current maximum from the quarter space-charge wavelength has been observed experimentally by Mihran[5] using a two-cavity klystron with a movable output cavity.

If the equations which have been presented are correct, they must reduce to Webster's ballistic equations which are known to be valid near the excitation plane where space charge effects are negligible. In Webster's theory the magnitude of the total r-f current density is given by

$$\left|\frac{J_z}{J_o}\right| = 2 \sum_n J_n\left(n \frac{\alpha}{2} \beta_e z\right) = 2 \sum_n J_n(nx) \quad . \tag{89}$$

Since space charge effects must be negligible, this requires that $\beta_{q1} z << \frac{\pi}{2}$, that is, the region under consideration is one in which electron bunches have not yet formed. At this point $\sin\beta_{q1} z \cong \beta_{q1} z = \frac{\omega_{q1}}{u_o} z$, and the magnitude of Equation (85) is then

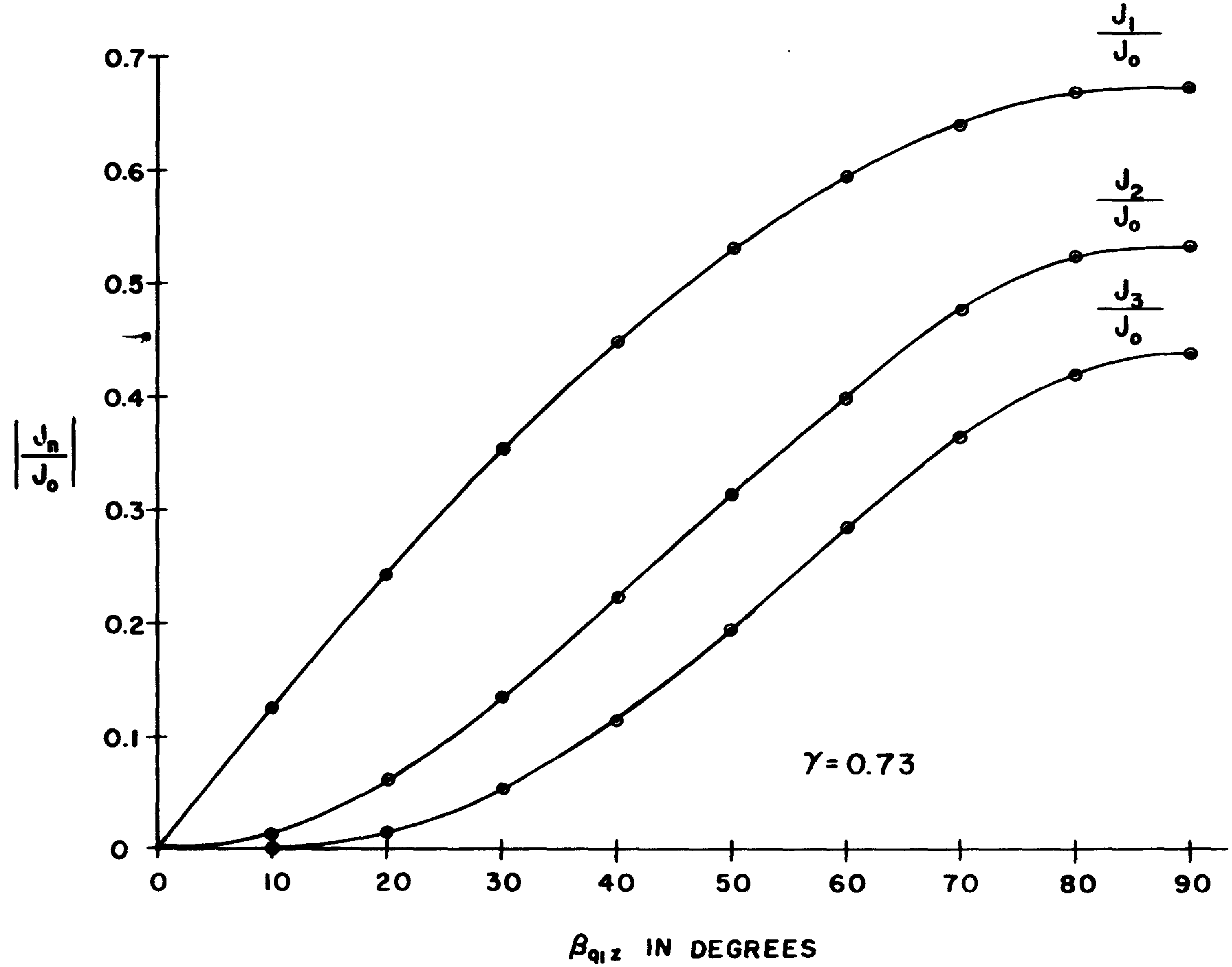

Figure 3. Current Density versus Distance $\gamma = 0.73$.

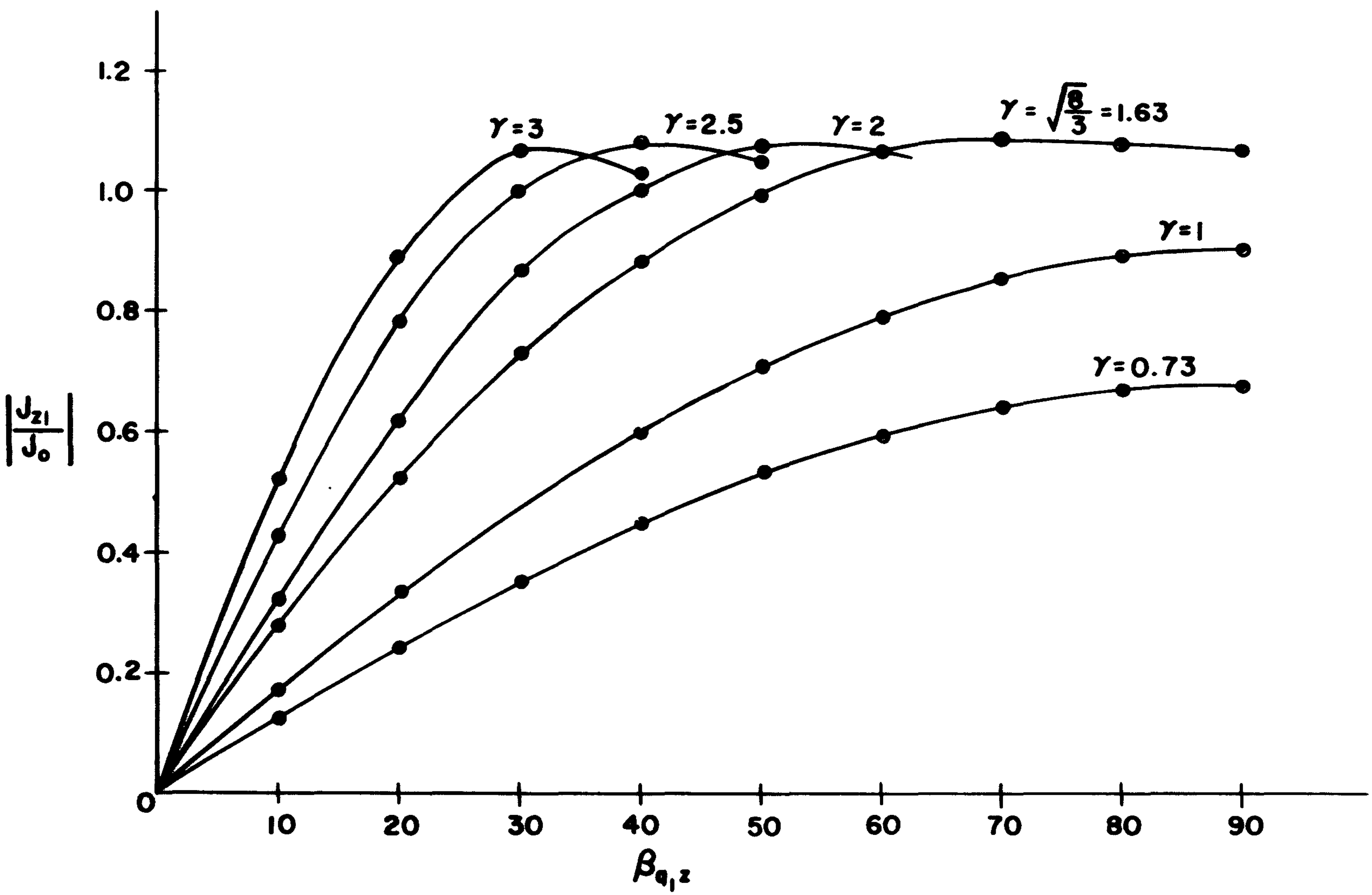

Figure 4. Fundamental Current Density versus Distance.

$$\left|\frac{J_z}{J_o}\right| = x + x^2 - \frac{1}{8}x^3 + \frac{9}{8}x^3 \quad . \tag{90}$$

The amplitudes of the current densities of frequencies $n\omega$ are given by

$$n = 1 : \quad \left|\frac{J_{z1}}{J_o}\right| = x - \frac{1}{8}x^3 \quad , \tag{91a}$$

$$n = 2 : \quad \left|\frac{J_{z2}}{J_o}\right| = x^2 \quad , \tag{91b}$$

$$n = 3 : \quad \left|\frac{J_{z3}}{J_o}\right| = \frac{9}{8}x^3 \quad . \tag{91c}$$

From Equation (89), by developing the Bessel functions into a power series for small arguments,

$$\left|\frac{J_{zn}}{J_o}\right| = \frac{2\left(\frac{nx}{2}\right)^n}{n!}\left(1 - \frac{\left(\frac{nx}{2}\right)^2}{n+1} + \ldots\right) \quad . \tag{92}$$

As expected, Equations (91) and (92) are in good agreement.

If the next term in the Bessel function expansion is introduced, Equation (91a) would become

$$\left|\frac{J_{z1}}{J_o}\right| = x - \frac{1}{8}x^3 + \frac{1}{192}x^5 - \ldots \quad . \tag{93}$$

This indicates that if higher than third order harmonics had been considered, the shift in the current maximum with increasing γ would be accompanied by an increase in amplitude rather than the slight decrease shown in Figure 4. This increase was present in the experimental observations of Mihran[5] mentioned previously.

If values of gap current are desired, they may be obtained by multiplying the harmonic current densities by the beam cross-sectional area S and by an average of $J_o^2(\beta_{tn}r)$ over the cross-sectional area. That is

$$I_n = S\left[J_o^2(\beta_{tn} r_{ep}) + J_1^2(\beta_{tn} r_{ep})\right] J_{zn} \quad . \tag{93}$$

Output gap voltages are then given by the product of harmonic gap currents and harmonic impedance where the impedance in this case must include the effects of beam loading.

CONCLUSIONS AND RECOMMENDATIONS

The analytic space charge wave solution shows that the harmonic densities are made up of several components. Each harmonic has a small amplitude component which has a distribution in space controlled by the harmonic reduced plasma phase constant. The main components however, result from the fundamental frequency beam disturbance and therefore have spatial distributions controlled by the fundamental reduced plasma phase constant. In contrast to Webster's ballistic theory, the total amplitudes are related directly to initial harmonic velocity modulation. This suggests that harmonic suppression may be accomplished by cancellation of harmonic velocity modulation at the excitation plane. The magnitudes of the harmonics can be accurately predicted up to a normalized drive level given by $\gamma = 0.736$ and the fundamental up to $\gamma = 2.8$ where γ is the ratio of operating to fundamental reduced plasma frequency multiplied by the ratio of one-half the equivalent infinitesimal gap r-f voltage to the beam voltage. Choice of γ determines the optimum drift tube length for maximum fundamental output.

Examination of Figure 3 shows that, at $\gamma = 0.73$, the maximums of all current density components occur at the quarter space-charge wavelength, and a decrease in harmonic level can be obtained only at the expense of the fundamental. At higher signal evels, Figure 4 shows that the fundamental maximum occurs at increasingly shorter drift lengths, while the harmonics exhibit dips in the region of the new fundamental maximum. It appears that parametric energy coupling is taking place at some multiple or multiples of the plasma frequency and that the ratio of harmonic output to fundamental output can be decreased with an increase in fundamental, by shortening the drift length between cavities at some large drive. The ultimate results of using a drive of $\gamma = 1.63$, for example, and a drift length $\beta_{ql} z = 70^{\circ}$ must be found by solving the large signal equations in a computer.

The good agreement between theoretical prediction of fundamental current density behavior and the experimental evidence of Mihran[5] suggests that a fifth order theory, which would give second and third harmonic saturation terms, would be of even greater usefulness in controlling spurious harmonics. Magnitude and direction of necessary changes in parameters could be pinpointed right up to the region of electron overtaking thereby saving valuable computer time.

APPENDIX A. SMALL SIGNAL COMPUTER RESULTS FOR THE SAL-36

The small signal equations were used to solve for displacements and velocities in the SAL-36 klystron amplifier.[6] To accomplish this, an IBM-650 digital computer was used to solve the systems of equations S1 and S2. The Milne method of numerical integration was used and an error of one per cent between "Predictor" and "Corrector" was tolerated. In using this method, all initial variational displacements and velocities were assumed to be zero, that is, $v = p = w = q = 0$.

Electric field configuration and limits were determined from low-frequency measurements made in an electrolytic tank.[7] Verification at high frequencies was obtained by using perturbation techniques.[8] Polynomial approximations to the experimental curves were found by employing the method of least squares and are

$$A(z) = 0.6000 - 2.1413 \times 10^2 z^2 + 2.7524 \times 10^4 z^4 - 1.0415 \times 10^6 z^6$$

$$B(z) = 0.1260 - 2.9369 \times 10^2 z^2 + 1.3899 \times 10^5 z^4 - 2.5014 \times 10^7 z^6 + 1.5467 \times 10^9 z^8$$

The field strength falls essentially to zero at $z = -1.65\,d$ and $z = 1.65\,d$ where $z = 0$ at the center of the cavity.

Solutions were begun at $z = -1.65\,d$ and carried in steps of 0.001 meters to a point one wavelength beyond the circuit field limit. The program used is shown in Figure 5 and curves of results are given in Figures 6 and 7. As mentioned in the text, the results must be modal analyzed before they can be used as initial conditions in the large signal case.

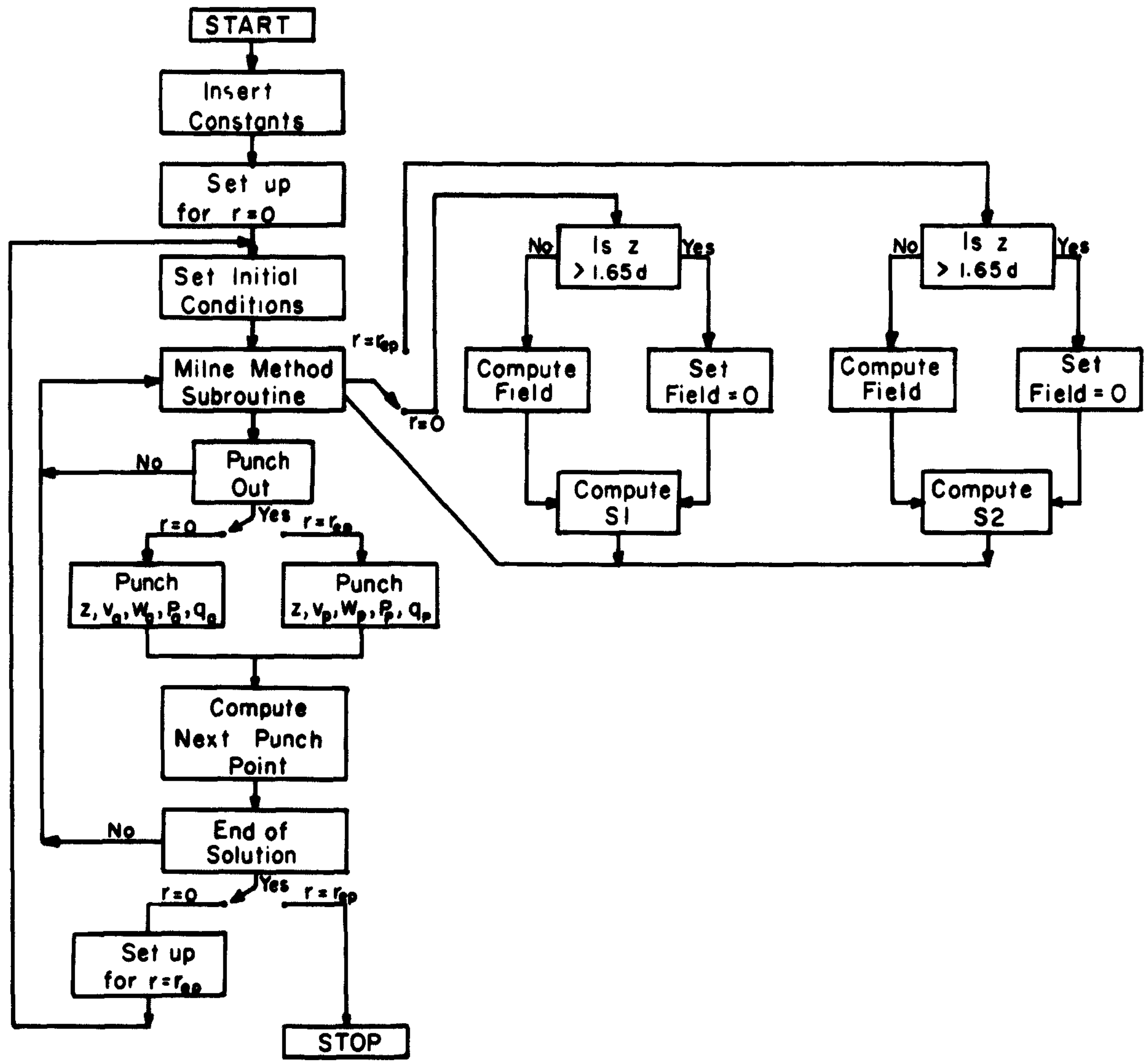

Figure 5. Small Signal Computer Program.

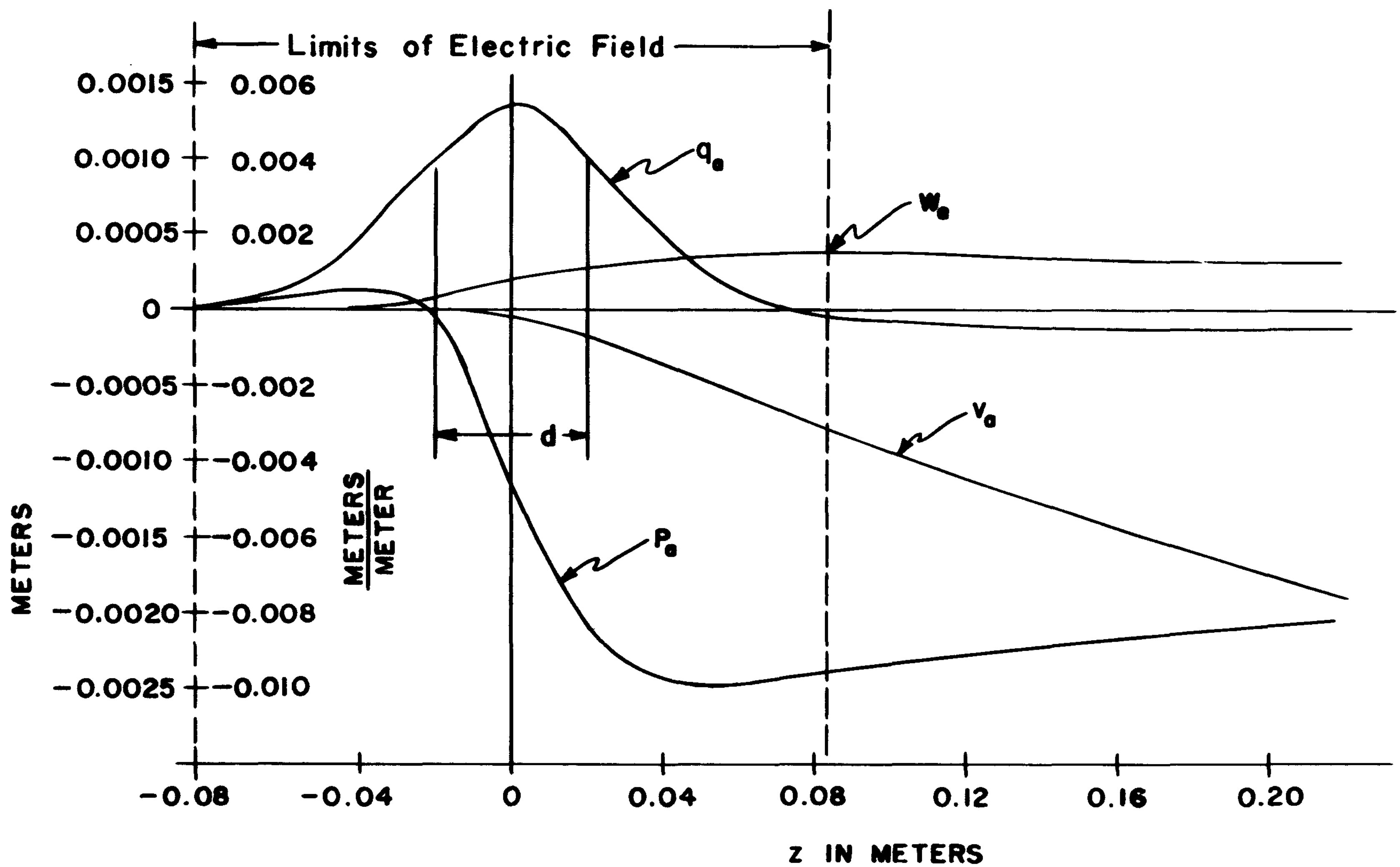

Figure 6. Computer Results for SAL-36 at r = 0.

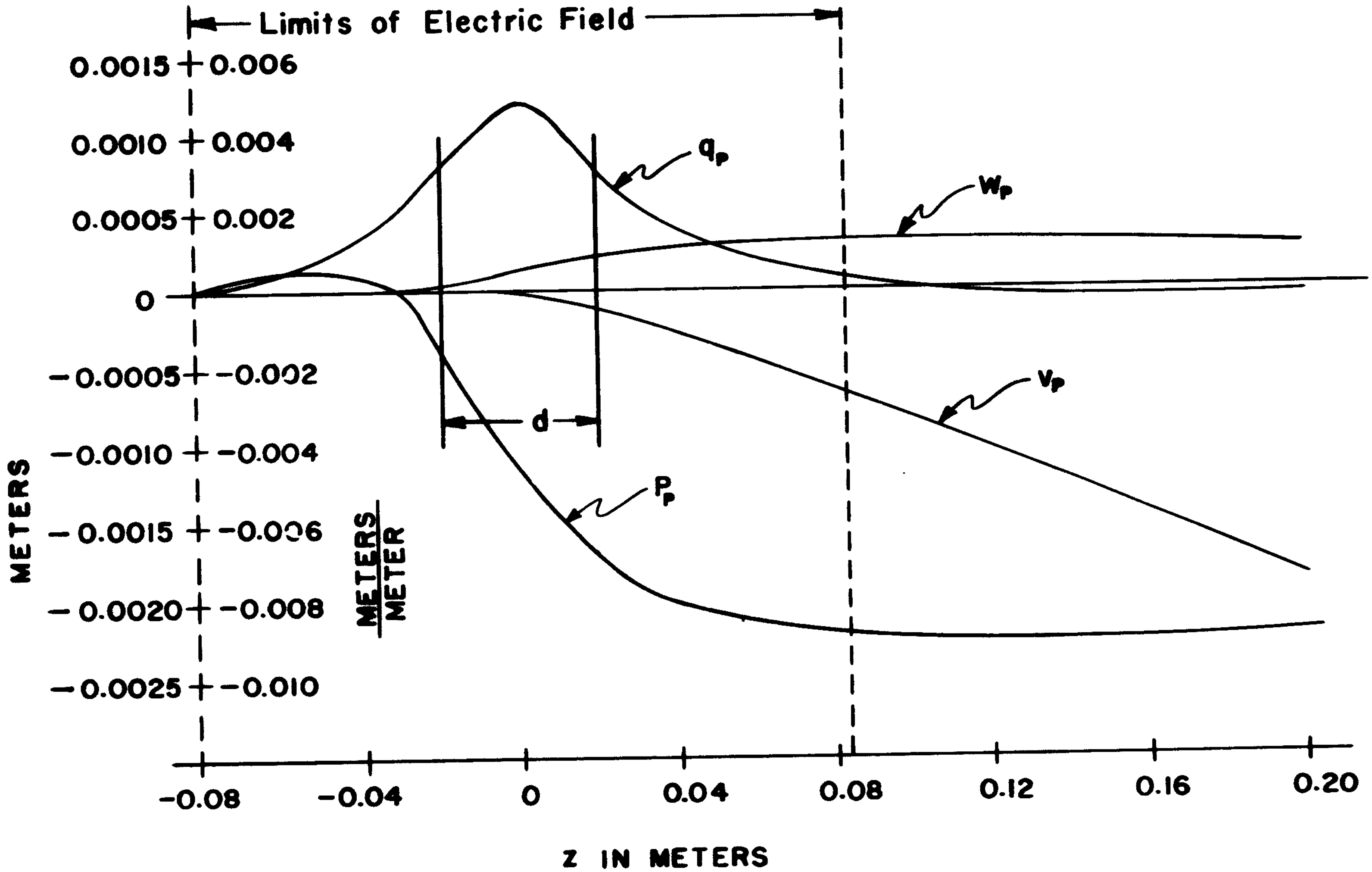

Figure 7. Computer Results for SAL-36 at $r = r_{ep}$.

Constants required to obtain solutions are

$$-\frac{e}{mu_o^2} = -3.4873 \text{ volt}^{-1}$$

$$-\frac{\omega_{qz}^2}{u_o^2} = -5.4277 \text{ meter}^{-1}$$

$$\gamma = \cos(\beta_e z + \phi) = \cos 31.2\,z \quad (\phi = 0 \text{ for first cavity})$$

$$\epsilon = \sin(\beta_e z + \phi) = \sin 31.2\,z \quad (\phi = 0 \text{ for first cavity})$$

$$|V| = 3 \text{ X } 10^3 \text{ volts}$$

$$d = 5.08 \text{ X } 10^{-2} \text{ meters}$$

APPENDIX B. LARGE SIGNAL COMPUTER PROGRAM FOR THE SAL-36

This Appendix has been added to show, in some detail, a practical application of the large signal equations up to the point of actually finding solutions using a digital computer. In this example a wavelength in ωt is broken up into twenty electron discs, each disc being eighteen degrees wide (θ_o = 18 degrees). The differential equations then apply to the i th disc and i varies from zero to twenty. The modified versions of the required equations are then:

$$(1 + \tilde{W}_i) \frac{\partial \tilde{W}_i}{\partial Y} = - \frac{\tilde{E}_{czi}}{2V_o\beta_e} - \frac{\tilde{E}_{szi}}{2V_o\beta_e} \quad , \tag{26B}$$

$$- \frac{\partial \theta_i}{\partial Y} = \frac{\tilde{W}_i}{1 + \tilde{W}_i} \quad . \tag{29B}$$

$$\text{Real } \frac{dI_n}{dY} = - \frac{i_o \left[J_o^2(\beta_{tn} r_{ep}) + J_1^2(\beta_{tn} r_{ep})\right] \mu'_{zn}(Y)}{\pi \beta_e d} \Delta\theta_o$$

$$\sum_{i=1}^{20} \cos(n\theta_i) \cos(nY) - \sin(n\theta_i) \sin(nY) \quad ,$$

$$\text{Imag } \frac{dI_n}{dY} = \frac{i_o \left[J_o^2(\beta_{tn} r_{ep}) + J_1^2(\beta_{tn} r_{ep})\right] \mu'_{zn}(Y)}{\pi \beta_e d} \Delta\theta_o$$

$$\sum_{i=1}^{20} \sin(n\theta_i) \cos(nY) + \cos(n\theta_i) \sin(nY) \quad . \tag{52B}$$

The equation numbers given here refer directly to the derived equations in the text. Thus, Equation (52B) is the modified version of Equation (52) and has been broken up into real and imaginary parts to facilitate solution.

In Equation (26B)

$$\tilde{E}_{czi} = \sum_{n=1}^{3} \frac{\mu'_{zn}(Y)\ |V_n|}{d} \cos(\phi_n + nY + n\theta_i) \ . \qquad (32B)$$

The constants in this equation are n and d, and Y is the independent variable generated by the computer; $\mu'_{zn}(Y)$ is determined from Equation (39) and the values of A(z) and B(z) are given in Appendix A, remembering that $Y = \beta_e z$ can be used to convert to A(Y) and B(Y). In all cavities except the first, an estimate of $|V_n|$ and ϕ_n must be made, and convergence to the known harmonic impedance Z_n is required as explained in the text.

Evaluation of $\tilde{E}_{szi}$ is slightly more complicated. In modified form,

$$\tilde{E}_{szi} = -\frac{\omega_p^2}{\omega^2}\ 2V_o\beta_e \sum_{j=1}^{20} \left[F(\theta_j - \theta_i) + \frac{\tilde{W}_j}{1+\tilde{W}_j}\ G(\theta_j - \theta_i)\right] \Delta\theta_j \ . \qquad (49B)$$

In the lossless case to be considered here, δ_n and σ_n are real and the expressions for $F(\theta_j - \theta_i)$ and $G(\theta_j - \theta_i)$ become

$$F(\theta_j - \theta_i) = \frac{1}{\pi} \sum_{n=1}^{3} \frac{\delta_n^2}{n} \sin n(\theta_j - \theta_i) \quad , \qquad (44B)$$

$$G(\theta_j - \theta_i) = \frac{1}{\pi} \sum_{n=1}^{3} \frac{\sigma_n^2}{n} \sin n(\theta_j - \theta_i) \quad ; \qquad (49B)$$

δ_n^2 is a function which has a maximum value of unity which it approaches at extreme high-frequency operation. When $\delta_n^2 = 1$, the F function becomes a sawtooth which can be evaluated by direct summation. Since δ_n^2 is an increasing function of n, it is much easier then to make a summation of the series $(1 - \delta_n^2)$ which represents the difference between the actual function and the linear sawtooth function. Symbolically,

$$\sum_n \frac{\delta_n^2}{n} \sin n(\theta_j - \theta_i) = \sum_n \frac{1}{n} \sin n(\theta_j - \theta_i) - \sum_n \frac{(1 - \delta_n^2)}{n} \sin n(\theta_j - \theta_i) .$$

For the particular case of the SAL-36, the summation was carried out using twenty terms and the resulting function $F(\theta_j - \theta_i)$ is plotted in Figure 8.

The series representing $G(\theta_j - \theta_i)$ converges rapidly as it stands and a plot of this function, after summing over twenty terms, is also shown in Figure 8.

Polynomial approximations to the curves in Figure 8 were written using the method of least squares. Finite disc widths were also accounted for in arriving at

$$F(\theta_j - \theta_i) = \sum_{k=1,3,5} a_k \left[(\theta_j - \theta_i)\, 20\right]^k \quad \text{for} \quad \left|(\theta_j - \theta_i)\right| < \frac{2\pi}{40} ,$$

$$F(\theta_j - \theta_i) = \sum_{k=1,3,5} a_k \left[\left(\pi - \left|(\theta_j - \theta_i)\right|\right) \frac{2\pi}{19}\right]^k \quad \text{for} \quad \left|(\theta_j - \theta_i)\right| > \frac{2\pi}{40} ,$$

$$G(\theta_j - \theta_i) = \sum_{k=1,3,5} b_k \left[(\theta_j - \theta_i)\, 20\right]^k \quad \text{for} \quad \left|(\theta_j - \theta_i)\right| < \frac{2\pi}{40} ,$$

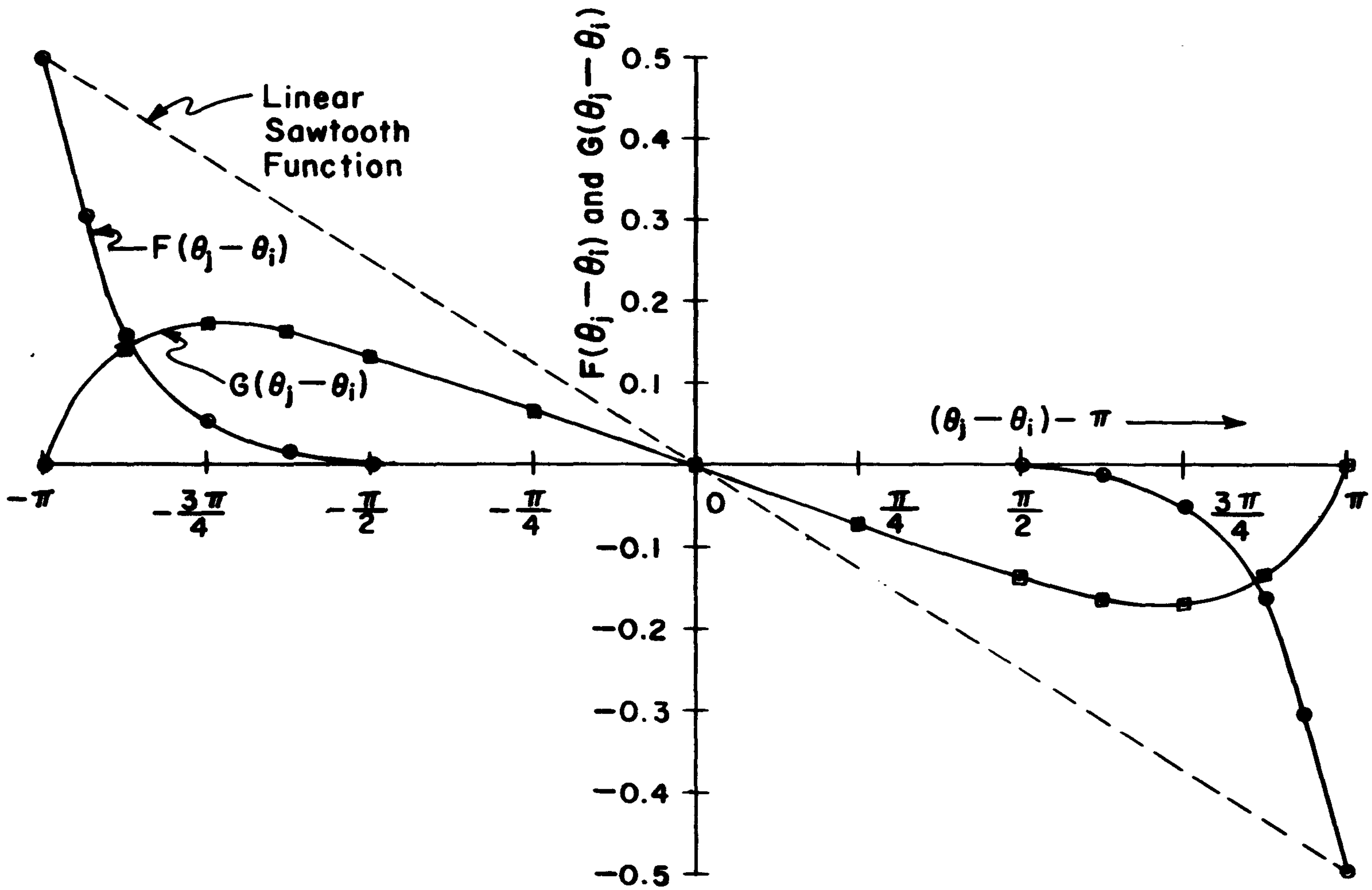

Figure 8. $F(\theta_j - \theta_i)$ and $G(\theta_j - \theta_i)$ as Functions of $(\theta_j - \theta_i) - \pi$ for the SAL-36 Klystron Amplifier.

$$G(\theta_j-\theta_i) = \sum_{k=1,3,5} b_k \left[\left(\pi - \left|(\theta_j-\theta_i)\right|\right) \frac{20}{19}\right]^k \quad \text{for} \quad \left|(\theta_j-\theta_i)\right| > \frac{2\pi}{40} \quad .$$

All other terms required to solve the equations are constants of the tube and its operating conditions, and it remains now to determine the initial conditions from the small signal solutions.

Since only the fundamental was considered in the linear portion of the tube, the constants K_{1n} and K_{2n} of Equation (38), become K_{11} and K_{21}. The small signal computations have given

$$\frac{P_z}{\rho_o} = \left[v(z) + jw(z)\right] e^{j(\omega t - \beta_e z)} \quad ,$$

$$\frac{u_z}{u_o} = \left[p(z) + jq(z)\right] e^{j(\omega t - \beta_e z)} \quad .$$

Realizing that $\tilde{\theta} = \beta_e \frac{P_z}{\rho_o}$, $\frac{\tilde{u}_z}{u_o} = \tilde{W}$, and $\theta_o = \theta + \tilde{\theta}$, the small signal results combined with Equation (17) and the constants K_{11} and K_{21} of the modal analysis result in initial conditions given by

$$\theta_o = \theta + \beta_e \Big\{K_{11} \left[v_a(z) \cos\theta - w_a(z) \sin\theta\right]$$
$$+ K_{21} \left[(v_p(z) \cos\theta - w_p(z) \sin\theta) - (v_a(z) \cos\theta - w_a(z) \sin\theta)\right]\Big\} \quad ,$$

$$\tilde{W} = K_{11} \left[p_a(z) \cos\theta - q_a(z) \sin\theta\right]$$
$$+ K_{21} \left[(p_p(z) \cos\theta - q_p(z) \sin\theta) - (p_a(z) \cos\theta - q_a(z) \sin\theta)\right] \quad .$$

These are evaluated at the arbitrary starting point of $z = 0.2847$ meters which is normalized to $Y = 8.883$ radians.

Summary of Procedure:

The small signal equations were solved from the limit of the circuit electric field in the first cacity to the point $Y = 8.883$ radians. At this point they were modal analyzed and became initial conditions for the large signal solution which is to be carried through the remainder of the first region to the limits of the circuit electric field in the second cavity. An initial estimate of induced cavity voltage (magnitude and phase) is made and inserted into the equations. The solution then proceeds through the circuit field region and harmonic cavity impedances are computed from the results. If the computed impedances do not agree with the known impedances, new estimates of voltage are made and the solution is again run through the circuit field region of the second cavity. This continues until the impedances agree. At this time the solution can be continued through the second drift region until the circuit field limits of the third cavity are reached. New estimates of voltage are then made and the converging procedure followed again. The final value of I_n determined in this way is the current which flows in the output circuit. Of course, the entire process could be continued for klystrons having more then three cavities.

A schematic of the SAL-36 is shown in Figure 9 and a large signal computer program is presented in Figure 10.

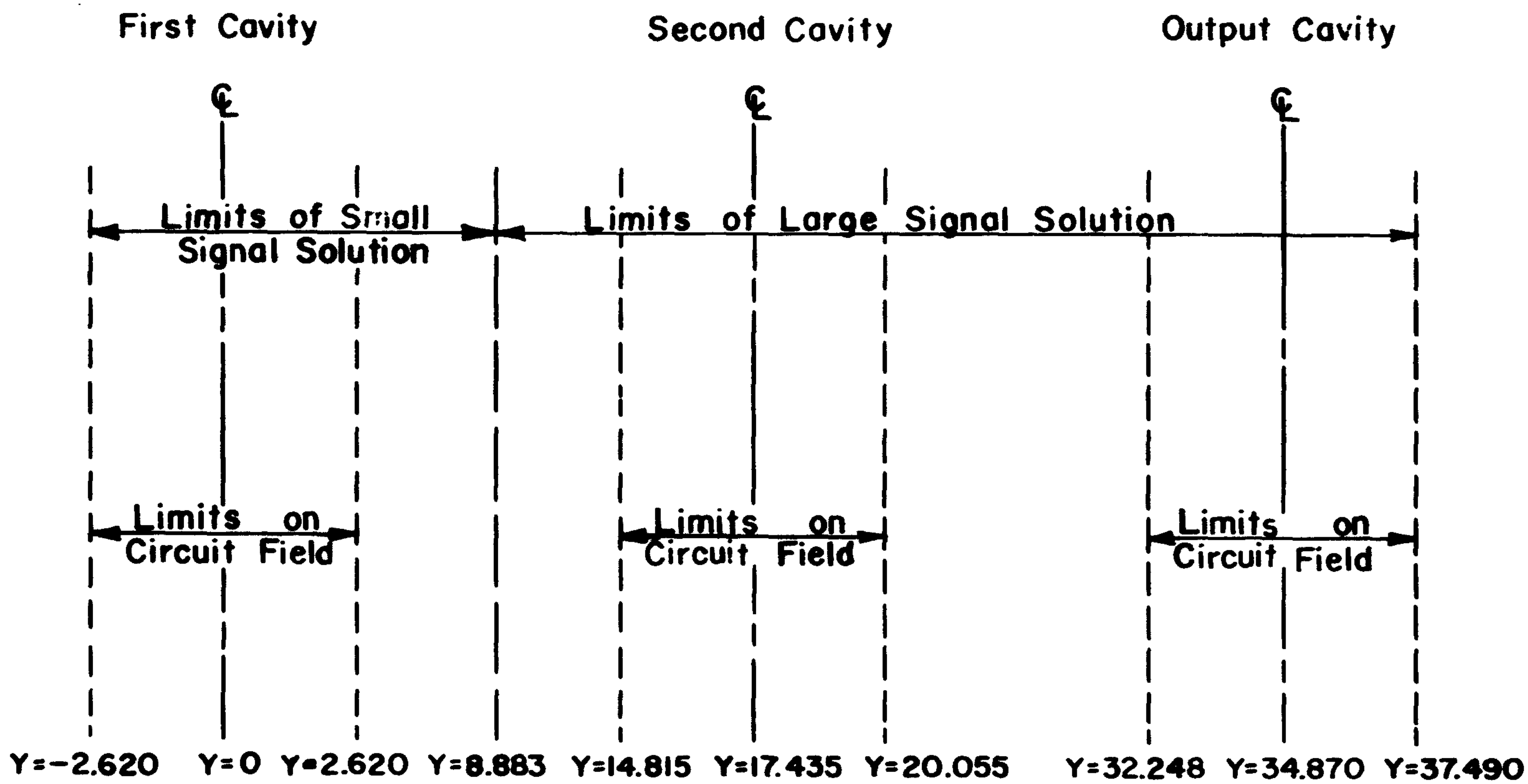

Figure 9. Schematic Diagram of the SAL-36 Klystron Amplifier.

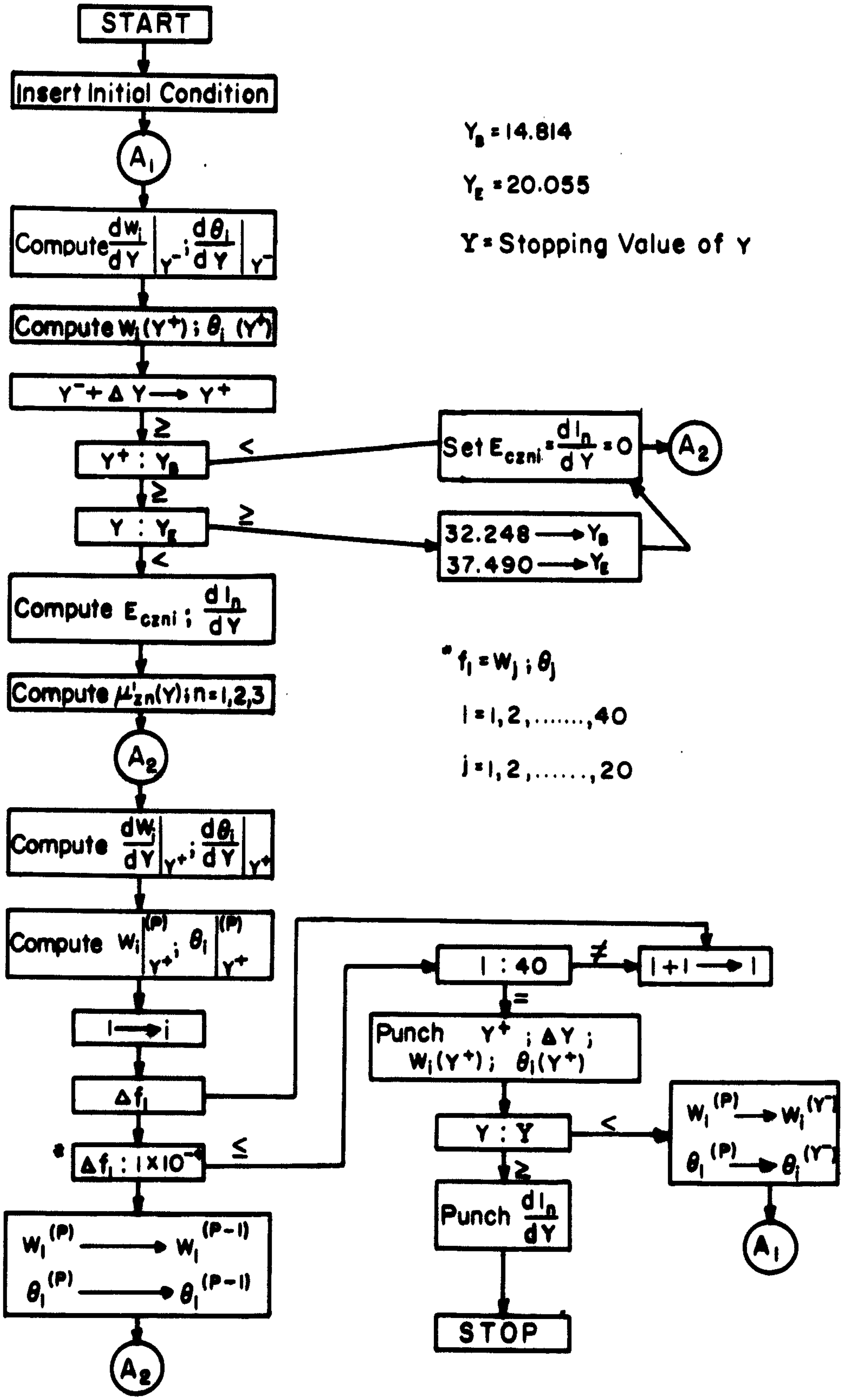

Figure 10. Large Signal Computer Program for the SAL-36.

The necessary constants and initial conditions are

$$V_o = 98.3 \times 10^3 \text{ volts}$$

$$\omega = 5.275 \times 10^9 \text{ second}^{-1}$$

$$u_o = 1.69 \times 10^8 \text{ meter second}^{-1}$$

$$\rho_o = 1.70 \times 10^{-4} \text{ coulomb meter}^{-3}$$

$$\beta_e = 31.2 \text{ meter}^{-1}$$

$$\omega_p = 1.645 \times 10^9 \text{ second}^{-1}$$

$$d = 5.08 \times 10^{-2} \text{ meter}$$

$$r_{ep} = 1.778 \times 10^{-2} \text{ meter}$$

$$-i_o = 28.3 \text{ ampere}$$

$$\Delta\theta_o = \frac{2\pi}{20} \text{ radian}$$

$$\beta_{tn} = 65.5 \text{ meter}^{-1} \qquad n = 1$$

$$79.1 \text{ meter}^{-1} \qquad n = 2$$

$$87.5 \text{ meter}^{-1} \qquad n = 3$$

$$K_{1n} = 1.175 \qquad n = 1$$

$$= 1.253 \qquad n = 2$$

$$= 1.33 \qquad n = 3$$

K_{2n}	=	2.22	n = 1	
	=	1.945	n = 2	
	=	1.67	n = 3	
a_k	=	-0.06085	k = 1	
	=	+0.03074	k = 2	
	=	-0.00411	k = 3	
b_k	=	-0.04901	k = 1	
	=	-0.01745	k = 2	
	=	+0.00227	k = 3	
V_n	=	3.909×10^4 volts	n = 1	Initial estimate
	=	0	n = 2, 3	
ϕ_n	=	-9.82	n = 1	Initial estimate
	=	0	n = 2, 3	
δ_n	=	0.310	n = 1	
	=	0.545	n = 2	
	=	0.690	n = 3	
σ_n^2	=	0.488	n = 1	
	=	0.345	n = 2	
	=	0.200	n = 3	

$$J_o^2(\beta_{tn} r_{ep}) + J_1^2(\beta_{tn} r_{ep}) = 0.7135 \qquad n = 1$$

$$= 0.606 \qquad n = 2$$

$$= 0.548 \qquad n = 3$$

$$\mu'_{z1}(Y) = 0.9847 - 0.9283(Y-17.435)^2 + 6.670 \times 10^{-2}(Y-17.435)^4$$
$$- 7.348 \times 10^{-3}(Y-17.435)^6 + 3.825 \times 10^{-3}(Y-17.435)^8 \quad ,$$

$$\mu'_{z2}(Y) = 0.9969 - 0.8625(Y-17.435)^2 + 6.493 \times 10^{-2}(Y-17.435)^4$$
$$- 6.690 \times 10^{-3}(Y-17.435)^6 + 3.351 \times 10^{-3}(Y-17.435)^8 \quad ,$$

$$\mu'_{z3}(Y) = 1.0084 - 0.7964(Y-17.435)^2 + 6.314 \times 10^{-2}(Y-17.435)^4$$
$$- 6.031 \times 10^{-3}(Y-17.435)^6 + 2.877 \times 10^{-3}(Y-17.435)^8 \quad .$$

Initial Conditions

Disc Number	θ_o	θ	$\tilde{W}$
1	0	0.1239	-0.0118
2	0.3142	0.4224	-0.0108
3	0.6283	0.7243	-0.0087
4	0.9426	1.0175	-0.0060
5	1.2566	1.3038	-0.0029
6	1.5708	1.5900	0.0005
7	1.8849	1.8797	0.0036
8	2.1994	2.1625	0.0067
9	2.5132	2.4504	0.0092
10	2.8278	2.7489	0.0106
11	3.1416	3.0421	0.0117
12	3.4562	3.3563	0.0115
13	3.7698	3.6687	0.0101
14	4.0846	4.0108	0.0075
15	4.3981	4.3459	0.0042
16	4.7124	4.6984	-0.0002
17	5.0264	5.0527	-0.0042
18	5.3414	5.4035	-0.0077
19	5.6547	5.7299	-0.0102
20	5.9698	6.0685	-0.0117

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P. S. Bottorff

School of Electrical Engineering
CORNELL UNIVERSITY
Ithaca, New York

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ABSTRACT

This paper considers aspects of electron motion in cylindrical electron beams focused by a magnetic field and moving within a cylindrical drift tunnel. The motions of electrons in such beams are of particular interest in the study of the operation of klystron generators of microwave energy.

Fundamental equations for the motion of electrons in magnetically focused beams are developed and combined with the Maxwell equations. The solutions obtained are applied to the study of the conditions existing in perturbed magnetically focused beams. Small-signal solutions are developed for both relativistic and nonrelativistic cases. For the special case of the Brillouin beam modulated by a gridless gap, large-signal solutions are found. Equations suitable for the study of the modulation of cylindrical magnetically focused beams by gaps are presented.

Application of the large-signal solutions to the problem of harmonic generation in Brillouin beams is considered. It is found that it is possible to adjust the beam conditions so that harmonic generation will be enhanced. Application of the large-signal solutions to the problem of current saturation in klystrons is described, and methods of reducing the output of unwanted harmonics are considered.

INTRODUCTION

In recent years there has been considerable interest in the production of high-power microwave energy. In conjunction with this interest considerable attention has been devoted to the theoretical analysis of the operation of linear-beam microwave tubes at high-power levels. Much of the analysis has been devoted to the study of the behavior of linear electron beams excited by a high-level of modulation.

This paper considers the motions of electrons in magnetically focused beams. Much attention has been devoted to the small-signal analysis of perturbed beams focused by magnetic means, but thus far no consideration has been given to the large-signal behavior of such beams. A large-signal solution, if it could be obtained, would furnish us with a better understanding of the behavior of microwave generators operating at high-power levels. It is the fundamental aim of this paper to present an analysis of the large-signal behavior of the Brillouin beam and to state some applications of the large-signal analysis to the operation of microwave klystrons.

Basic to the analysis of the motions of electrons in beams is the combination of Newton's and Lorentz's laws in the equation of motion:

$$m \frac{d\underline{v}}{dt} = e(\underline{E} + \underline{v} \times \underline{B}) \quad . \tag{1}$$

In the Eulerian description of the motion, Equation (1) is rewritten in the form

$$\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot (\nabla \underline{v}) = \frac{e}{m} (\underline{E} + \underline{v} \times \underline{B}) \quad . \tag{2}$$

The velocity $\underline{v}$ has two components: (a) the d-c velocity produced by the focusing system used to align and accelerate the electrons after they leave the cathode, and (b) a component which will be produced by the modulating structure. We shall denote the d-c part by $\underline{v}_o$ and the part consisting of perturbations causes by the modulation by $\tilde{\underline{v}}$. We refer to $\underline{v}_o$ as the d-c velocity, since it does not normally vary with time.

We may rewrite Equation (2) as

$$\frac{\partial \tilde{\underline{v}}}{\partial t} + (\underline{v}_o + \tilde{\underline{v}}) \cdot [\nabla (\underline{v}_o + \tilde{\underline{v}})] = \eta (\underline{E} + \underline{v} \times \underline{B}) \quad , \tag{3}$$

where $\eta = e/m$. For the moment, let us assume that $\underline{v}_o$ has no spatial variations so that $\nabla \underline{v}_o = 0$. Then Equation (3) becomes:

$$\frac{\partial \tilde{\underline{v}}}{\partial t} + \underline{v}_o \cdot (\nabla \tilde{\underline{v}}) + \tilde{\underline{v}} \cdot (\nabla \tilde{\underline{v}}) = \eta (\underline{E} + \underline{v} \times \underline{B}) \quad . \tag{4}$$

If $|\underline{v}_o| >> |\underline{v}|$ then the last term in the left-hand side of Equation (4) may be neglected. This is the small-signal assumption, and is usually made in the study of the operation of a device at low-power levels.

Equation (4) involves two types of co-ordinates, the time, and the set of position co-ordinates used to fix the location of the electron in space. The first term on the left-hand side of Equation (4) accounts for the explicit change of velocity with change in time at a constant position. Of course, the electron will not remain at a constant position, and it is the purpose of the last two terms on the left-hand side of Equation (4) to account for changes in the velocity of the electron caused by dependence on the position co-ordinates. Change in the position co-ordinates may, of course, be caused either by the d-c velocity of the electrons or by the velocity resulting from the perturbations. If the disturbance is small, the change in position resulting from the velocity associated with the perturbations will be small compared to the change in position resulting from the d-c velocity, and the last term on the left-hand side of Equation (4) may be ignored.

However, if the beam is strongly excited, the last term on the left-hand side of Equation (4) becomes important and may no longer be neglected. The physical process that produces this effect may be described as follows: The excitation produces motions o' the electrons that are not uniform in space, and as a result, electric and magnetic fields are set up within the beam. For small values of excitation, the values of electric and magnetic fields are determined by the linearized form of Equation (4), the form with the last term on the left-hand side neglected. However, these electric and magnetic fields in the beam will stimulate further motions, and, if the values of excitation are large, these new motions must be accounted for. This may be done by using the nonlinearized form of Equation (4). Just as the equation of motion is nonlinear if the excitation is high, the beam also is nonlinear.

Some of the theoretical analyses of electron beams can now be considered in the light of the preceding discussion. The first theoretical analysis was performed by Webster[1] using a ballistic approach. Webster analyzed the motions of an electron beam excited by a gridded gap, neglecting the effects of space charge, and was able to deduce an expression for the efficiency of a klystron. His method, however, suffers from the defects that no space-charge effects are taken into account and that the electromagnetic fields outside the excitation region are ignored. Thus the results can validly be applied only to beams of very low density in which the electromagnetic effects are small in comparison with the ballistic effects.

In an effort to overcome the defects of the ballistic approach, Hahn[2] and Ramo[3] combined the linearized form of the equation of motion with the Maxwell equations. They found that the electron motions should have a wave-like character, periodic along the beam. Their solutions have been termed "space-charge waves."

Both these methods of analysis assumed that the electrons were confined to move in a direction parallel to the axis of the beam. This is approximately the case when the magnetic field used to focus the beam has very large values. In actual devices, however, the values used for the magnetic focusing field are close to the minimum possible value for stability of the beam, in order to save space and weight and to reduce the cost of the focusing system. In such cases, much of the electron motion will not be in a direction parallel to the axis of the beam. Therefore we must examine some of the methods used in the analysis of the properties of focusing systems and the characteristics of the high-frequency behavior of magnetically focused electron beams.

Conditions for the stable focusing of electron beams by magnetic fields were first investigated by Brillouin,[4] and subsequently by Wang,[5] Samuel,[6] Brewer,[7] and Dow.[8] Underlying these analyses is the assumption of laminar flow, which requires that the paths of the electrons not cross each other. This assumption is made necessary because the functions that describe the electron velocities must be restricted to being single-valued. Laminar flow is not achieved in practical beams, as has been demonstrated by Harker,[9] but there is no reason to believe that it is not a good approximation, if the thermal velocities of the electrons emitted from the cathode are small with respect to the accelerating potentials, which is usually the case.

It is also assumed that the electrons leave the cathode with zero potential energy and velocity. This assumption is also valid, and when taken with the assumption of laminar flow, implies similarity of electron paths as the electrons move along the beam.

The authors cited assumed that the electric and the magnetic fields associated with the focusing system were axially symmetric, and that the charge density was uniform at all points in the beam. Only careful cathode and gun design could approach these conditions in practice, and these last assumptions are probably more restrictive than those previously stated.

Wang[5] has shown that the angular frequency of rotation of electrons about the axis of the beam is strongly dependent on the amount of magnetic flux threading the cathode. If no magnetic flux threads the cathode, and if the other assumptions mentioned are valid, then we have the focusing conditions specified by Brillouin:[4]

1. All electrons in the beam will have an angular frequency equal to the Larmor frequency, ω_L, where $\omega_L = \eta B_o/2$, and B_o is the strength of the applied magnetic field.
2. The radial velocity is zero, the axial velocity is everywhere constant, and the charge density is uniform.

When the Brillouin conditions are satisfied, the value of B_o is the smallest that may be used for stable focusing. If there is magnetic flux threading the cathode, the value of B_o required to produce a stable beam will be larger than the Brillouin value and the surface of the beam will be rippled or "scalloped." Wang[5] has established conditions for stability in such beams.

Now that we have reviewed the analysis of magnetically focused beams in the unperturbed state, we shall briefly consider the analyses that have been made of magnetically focused electron beams excited by a modulating device.

The problem has been considered by Rigrod and Lewis,[10] Brewer,[11] Labus,[12] and Paschke.[13] Their methods are all essentially the same: The linearized form of Equation (4) is combined with the field equations to yield wave-like solutions. However, they differ considerably in the methods employed to calculate the phase constants of the waves, the difference centering around the problem of the boundary conditions to be used at the edge of the beam.

Figure 1 shows the model for an excited magnetically focused electron beam. The surface of the beam is rippled after excitation. The boundary conditions are that $\tilde{E}_z$ the axial, and $\tilde{E}_r$, the radial electric fields caused by the perturbation be continuous at the edge of the beam. However, the location of the edge of the beam is a function of the time, and it is difficult to develop a method of matching the electromagnetic fields at the constantly shifting boundary of the beam; therefore the beam is represented by the model shown in Figure 2. The beam is not rippled and the location of the edge is constant in time, with the ripples replaced by an equivalent surface charge density ρ_s, or an equivalent surface current $\underline{J}_s$. This replacement permits equating of the fields inside and outside of the beam at $r = b$.

The disagreement among the various authors centers around the method to be used in computing the surface current. Rigrod and Lewis[10] use only the ripples in the boundary layer region to calculate the surface

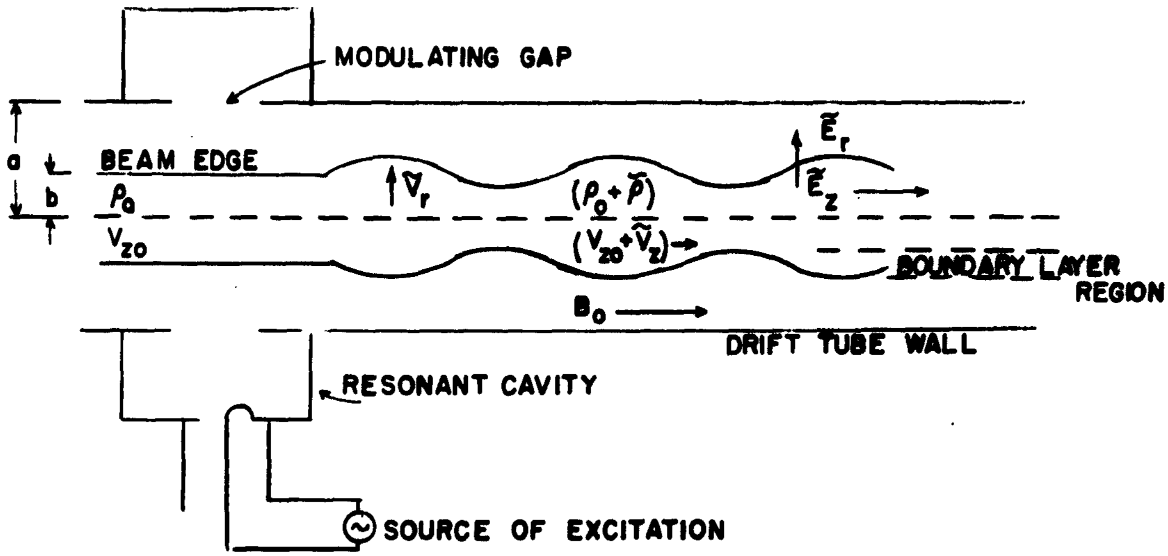

Figure 1. Model of Perturbed Electron Beam.

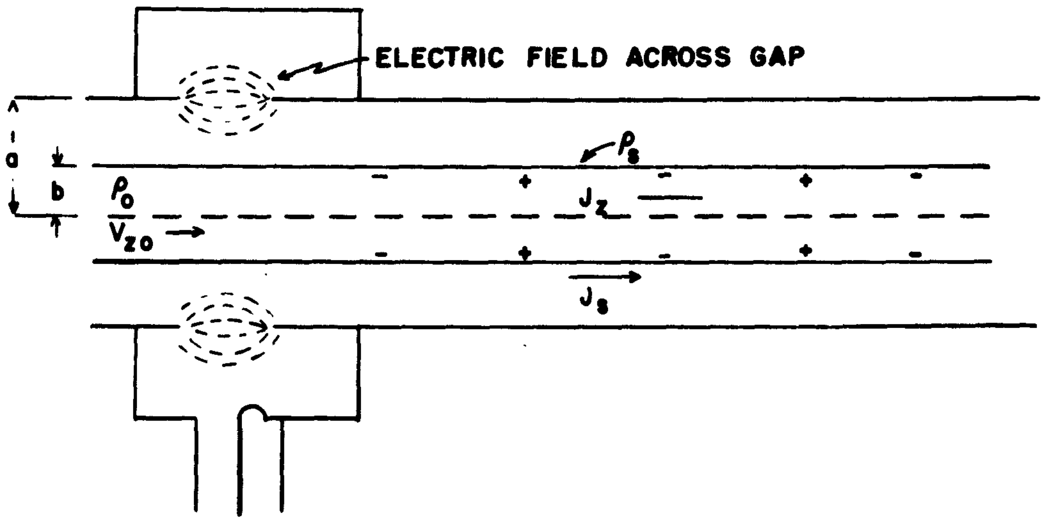

Figure 2. Replacement of Rippled Beam by Smooth Beam with Surface Charge.

current, while Labus[12] and Paschke[13] think that the ripples throughout the beam must be considered. In addition, Paschke uses a somewhat different formulation of the equations of motion from that used by other authors. In this paper the method of Rigrod and Lewis is used in formulating the boundary conditions.

We have so far reviewed only small-signal theories of electron motion. Work on the large-signal problem has thus far been restricted to the case of the beam in which all electrons are confined to move in a direction parallel to the axis of the beam. Studies of the large-signal behavior of the confined beam have been made by Paschke,[14, 15, 16] Romaine,[17] Blair,[18] and Engler.[19] The method used has been to combine Equation (1) with the Maxwell equations. The resulting nonlinear differential equation has then been solved by the method of successive approximation. Beams of both infinite and finite geometry have been studied.

The work just described has been restricted to the case of thin electron beams, where beams of finite geometry have been studied. Recently, Olving[20] has succeeded in deriving a nonlinear theory for the thick, cylindrical, finite beam. His results, when applied to a thin beam, reduce to those found by the other authors.

No work has appeared to date on a nonlinear theory of a magnetically focused beam. The remainder of this paper will be devoted primarily to the consideration of the development of a nonlinear theory of the Brillouin beam.

FUNDAMENTAL EQUATIONS AND ASSUMPTIONS FOR THE ANALYSIS OF THE ELECTRON MOTION

The following assumptions are basic to the analysis:

1. The electric and magnetic fields associated with the motion are axially symmetric.
2. The electron flow is laminar, and the initial velocity of the electron beam is uniform at all points in the beam. The charge density of the unperturbed beam is uniform at all points inside the beam and is zero outside the beam. The charge density of the unperturbed beam will be denoted by ρ_o.
3. The radial velocity of the electrons in the unperturbed beam is negligible with respect to the other velocities involved in the problem. (Thus any effects resulting from the scalloping of the d-c beam are not to be considered.) The axial velocity is small with respect to the velocity of light.
4. The beam moves within a perfectly conducting drift tunnel and is excited by a gridless gap in the tunnel. The drift region is free of any electric and magnetic fields caused by the beam itself, except for the field of the focusing magnet.
5. Thermal effects may be ignored.

The co-ordinate system used is cylindrical polar. Distances in the axial or z direction are measured from the center of the modulating gap. Radial or r-direction distances are measured from the central axis of the beam. Azimuthal or θ direction distances may be measured from any convenient reference. The co-ordinate system to be used is shown in Figure 3.

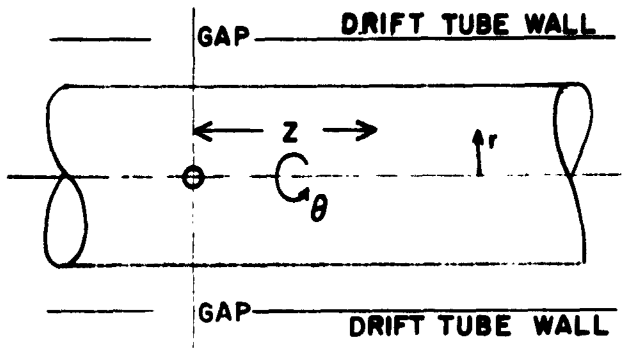

Figure 3. Description of Co-ordinate System Used in the Analysis.

Assumption 1 requires that $\partial/\partial\theta = 0$, which is reasonable, since the structure surrounding the beam, and the beam itself, are presumed to be axially symmetric. Assumptions 2 and 3 are never valid in real structures, but experimental work by Gilmour[21] has shown that beams with properties closely approximating the ideal behavior assumed may be produced. Assumptions 4 and 5 are justified in actual beams. Appendix B gives an analysis of motions in beams where the initial velocity v_{zo} is not much less than the velocity of light.

We are now ready to begin the analysis. In cylindrical co-ordinates, Equation (1), the equation of motion, may be written

$$\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r} = \eta(E_r + v_\theta B_z - v_z B)_\theta \quad , \qquad (5a)$$

$$\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + v_z \frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r} = \eta(E_\theta + v_z B_r - v_r B_z) \quad , \qquad (5b)$$

$$\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} = \eta(E_z + v_r B_\theta - v_\theta B_r) \quad . \qquad (5c)$$

The subscripts to the quantities in Equation (5) refer to the components in each direction.

We shall need one other set of basic relations, the Maxwell equations:

$$\nabla \cdot \underline{E} = \rho/\epsilon_o \quad , \tag{6}$$

$$\nabla \cdot \underline{B} = 0 \quad , \tag{7}$$

$$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t} \quad ; \tag{8}$$

or

$$\frac{\partial E_\theta}{\partial z} = \frac{\partial B_r}{\partial t} \quad , \tag{8a}$$

$$\frac{1}{r}\,\frac{\partial (rE_\theta)}{\partial r} = -\frac{\partial B_z}{\partial t} \quad , \tag{8b}$$

$$\frac{\partial E_z}{\partial r} - \frac{\partial E_r}{\partial z} = \frac{\partial B_\theta}{\partial t} \quad , \tag{8c}$$

and

$$\nabla \times \underline{B} = \mu_o\, \rho\underline{v} + \frac{1}{c^2}\,\frac{\partial E}{\partial t} \quad ; \tag{9}$$

or

$$-\frac{\partial B_\theta}{\partial z} = \frac{1}{c^2}\left\{ v_r\left[\frac{1}{r}\,\frac{\partial}{\partial r}\,(rE_r) + \frac{\partial E_z}{\partial z}\right] + \frac{\partial E_r}{\partial t}\right\} \quad , \tag{9a}$$

$$\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} = \frac{1}{c^2}\left\{ v_\theta\left[\frac{1}{r}\,\frac{\partial}{\partial r}\,(rE_r) + \frac{\partial E_z}{\partial z}\right] + \frac{\partial E_\theta}{\partial t}\right\} \quad , \tag{9b}$$

$$\frac{1}{r}\,\frac{\partial}{\partial r}\,rB_\theta = \frac{1}{c^2}\left\{ v_z\left[\frac{1}{r}\,\frac{\partial}{\partial r}\,(rE_r) + \frac{\partial E_z}{\partial z}\right] + \frac{\partial E_z}{\partial t}\right\} \quad . \tag{9c}$$

The two sets of equations can be seen to be related to each other, since the velocities appear in the Maxwell equations and the electromagnetic fields in the equations of motion.

The first step is to consider the motion of the unperturbed beam. The fields to be considered now are those caused only by the space charge of the unperturbed beam in the tunnel and the focusing magnetic field. As a consequence of symmetry and the assumption that there is no d-c axial electric field in the drift region, we have

$$E_{zo} = B_{ro} = E_{\theta o} = 0 \quad .$$

(The subscript o will be used to denote steady-state quantities.) A uniform magnetic field $\underline{B}_o = B_o \hat{z}$ is assumed to be present to focus the beam, where $\hat{z}$ is the unit vector in the z direction. Since $v_{ro} = 0$ by assumption, and v_{zo} is constant, the equations of motion reduce to

$$-r\omega_o^2 = \eta E_{ro} + \eta r \omega_o B_o \quad , \tag{10}$$

where ω_o is the angular frequency of rotation of the electrons about the axis of the beam. Maxwell's equations reduce to

$$\frac{1}{r} \frac{\partial (rE_{ro})}{\partial r} = \frac{\rho_o}{\epsilon_o} \quad , \tag{11a}$$

$$-\frac{\partial B_{zo}}{\partial r} = \mu_o \rho_o r\omega_o \quad , \tag{11b}$$

$$\frac{1}{r} \frac{\partial}{\partial r} rB_{\theta o} = \mu_o \rho_o v_{zo} \quad . \tag{11c}$$

Next let us define

$$\omega_p = \sqrt{\frac{\eta \rho_o}{\epsilon_o}} \quad . \tag{12}$$

In simple space-charge-wave analysis, it is found that ω_p is the frequency of oscillation of an excited infinite beam of electrons; then

$$E_{ro} = \frac{\omega_p^2}{2\eta} r \quad , \tag{13a}$$

$$B_{zo} = B_o - \mu_o \rho_o \omega_o r^2 \quad , \tag{13b}$$

$$B_{\theta o} = \frac{\omega_p^2 v_{zo} r}{2\eta c^2} \quad . \tag{13c}$$

In practical cases we may set

$$B_{zo} = B_o$$

and

$$B_{\theta o} = 0 \quad ,$$

since $v_{zo} \ll c$. Now define

$$\omega_L = \frac{\eta B_o}{2} \quad . \tag{14}$$

Combining Equation (10), (13), and (14), we get

$$\omega_o = -\omega_L (1 \pm k_m) \quad , \tag{15a}$$

where

$$k_m = \sqrt{1 - \frac{\omega_p^2}{2\omega_L^2}} \quad . \tag{15b}$$

For the Brillouin case $k_m = 0$. We shall be concerned only with cases for which $\omega_o = -\omega_L(1 + k_m)$.

Equations (15a) and (15b) express the conditions for equilibrium in a magnetically focused beam. Physically, Equation (15) expresses the conditions that space-charge, magnetic, and centripetal forces must add to zero.

The motions of electrons under the influence of a disturbance can now be considered. Let

$$v_r = \tilde{v}_r \quad , \tag{16a}$$

$$v_\theta = r\omega_o + \tilde{v}_\theta \quad , \tag{16b}$$

$$v_z = v_{zo} + \tilde{v}_z \quad . \tag{16c}$$

Equations (5) may then be written as

$$\frac{d\tilde{v}_r}{dt} + 2k_m\omega_L\tilde{v}_\theta + v_r\frac{\partial\tilde{v}_r}{\partial r} + v_z\frac{\partial\tilde{v}_r}{\partial z} - \frac{\tilde{v}_\theta^2}{r}$$

$$= \eta(\tilde{E}_r + \tilde{v}_\theta\tilde{B}_z + r\omega_o\tilde{B}_z - \tilde{v}_z\tilde{B}_\theta - v_{zo}\tilde{B}_\theta) \quad , \tag{17a}$$

$$\frac{d\tilde{v}_\theta}{dt} - 2k_m\omega_L\tilde{v}_r + \tilde{v}_r\frac{\partial\tilde{v}_\theta}{\partial r} + \tilde{v}_z\frac{\partial\tilde{v}_\theta}{\partial z} + \frac{\tilde{v}_r\tilde{v}_\theta}{r} = \eta(\tilde{E}_\theta + [(v_{zo} + \tilde{v}_z)\tilde{B}_r - \tilde{v}_r\tilde{B}_z] \quad , \tag{17b}$$

$$\frac{d\tilde{v}_z}{dt} + \tilde{v}_r\frac{\partial\tilde{v}_z}{\partial r} + \tilde{v}_z\frac{\partial\tilde{v}_z}{\partial z} = \eta[\tilde{E}_z + \tilde{v}_r\tilde{B}_\theta - (r\omega_o + \tilde{v}_\theta)\tilde{B}_r] \quad , \tag{17c}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v_{zo}\frac{\partial}{\partial z} \quad . \tag{18}$$

Besides the velocities and the electromagnetic fields, there is another vector variable quantity which is of interest. This is the polarization distance, which we define as the displacement of any element of charge dq from its position in the unperturbed beam under the influence of the perturbation. The displacement is, of course, caused by the perturbation.

Let r_o, θ_o, z_o be the co-ordinates of an electron before any disturbance takes place. Then

$$r = r_o + P_r(r, \theta, z, t) \quad , \tag{19a}$$

$$\theta = \theta_o + \omega_o t + \frac{P_\theta(r, \theta, z, t)}{r} \quad , \tag{19b}$$

$$z = z_o + v_{zo}t + P_z(r, \theta, z, t) \quad . \tag{19c}$$

At a time $t = \Delta t$,

$$r + \Delta r = r_o + P_r(r + \Delta r, \theta + \Delta\theta, z + \Delta z, t + \Delta t) \quad , \tag{20a}$$

$$\theta + \Delta\theta = \theta_o + \omega_o(t + \Delta t) + \frac{P_o(r + \Delta r, \theta + \Delta\theta, z + \Delta z, t + \Delta t)}{r + \Delta r} \quad , \tag{20b}$$

$$z + \Delta z = z_o + v_{zo}(t + \Delta t) + P_z(r + \Delta r, \theta + \Delta\theta, z + \Delta z, t + \Delta t) \quad . \tag{20c}$$

Substracting Equations (19) from (20) and letting Δt approach zero, we find:

$$v_r = \frac{\partial P_r}{\partial t} + (\underline{v} \cdot \nabla)\, P_r \quad , \tag{21a}$$

$$v_z = v_{zo} + \frac{\partial P_z}{\partial t} + (\underline{v} \cdot \nabla) P_z \quad , \tag{21b}$$

$$v_\theta = r\omega_o + \frac{\partial P_\theta}{\partial t} + (\underline{v} \cdot \nabla) P_\theta - \frac{P_\theta}{r} v_r \quad . \tag{21c}$$

We may now solve Equations (21) for the velocities:

$$v_r = \frac{\frac{\partial P_r}{\partial t}\left(1 - \frac{\partial P_z}{\partial z}\right) + \left(v_{zo} + \frac{\partial P_z}{\partial t}\right)\left(\frac{\partial P_r}{\partial z}\right)}{\left(1 - \frac{\partial P_r}{\partial r}\right)\left(1 - \frac{\partial P_z}{\partial z}\right) - \frac{\partial P_r}{\partial z}\frac{\partial P_z}{\partial r}} \quad , \tag{22a}$$

$$v_z = \frac{\left(v_{zo} + \frac{\partial P_z}{\partial t}\right)\left(1 - \frac{\partial P_z}{\partial r}\right) + \frac{\partial P_r}{\partial t}\frac{\partial P_z}{\partial r}}{\left(1 - \frac{\partial P_r}{\partial r}\right)\left(1 - \frac{\partial P_z}{\partial z}\right) - \frac{\partial P_r}{\partial z}\frac{\partial P_z}{\partial r}} \quad , \tag{22b}$$

$$v_\theta = r\omega_o + \frac{\partial P_\theta}{\partial t} + \left\{\left(v_{zo} + \frac{\partial P_z}{\partial t}\right)\left[\left(1 - \frac{\partial P_r}{\partial t}\right)\left(\frac{\partial P_\theta}{\partial t}\right) + \frac{\partial P_r}{\partial t}\left(\frac{\partial P_\theta}{\partial r} - \frac{P_\theta}{r}\right)\right]\right.$$

$$\left. + \frac{\partial P_r}{\partial t}\left[\frac{\partial P_\theta}{\partial z}\frac{\partial P_z}{\partial r} + \left(\frac{\partial P_\theta}{\partial r} - \frac{P_\theta}{r}\right)\left(1 - \frac{\partial P_z}{\partial z}\right)\right]\right\}$$

$$\left\{\left[\left(1 - \frac{\partial P_r}{\partial r}\right)\left(1 - \frac{\partial P_z}{\partial z}\right) - \frac{\partial P_z}{\partial r}\frac{\partial P_r}{\partial z}\right]^{-1}\right\} \quad . \tag{22c}$$

There are now two possible methods of obtaining a nonlinear differential equation that can be used to describe the properties of the motion.

We can work with the velocities directly or we can substitute Equation (22) into Equation (17) and use the polarizations as the basic variables in the problem. For our purposes, we shall find that it is easier to work directly with the velocities. Appendix C gives the nonlinear equations of motion with the polarizations as basic variables.

In order to solve Equations (17) in conjunction with the field equations, it will be necessary to assume that a solution is possible by the method of successive approximation. We assume that we can write

$$\underline{\tilde{v}} = \underline{\tilde{v}}_1 + \underline{\tilde{v}}_2 + \underline{\tilde{v}}_3 + \dots \quad , \tag{23a}$$

$$\underline{\tilde{E}} = \underline{\tilde{E}}_1 + \underline{\tilde{E}}_2 + \underline{\tilde{E}}_3 + \dots \quad , \tag{23b}$$

$$\underline{\tilde{B}} = \underline{\tilde{B}}_1 + \underline{\tilde{B}}_2 + \underline{\tilde{B}}_3 + \dots \quad . \tag{23c}$$

Each of the terms in Equation (23) is presumed to be less in magnitude than the one preceding it, i.e.

$$|\underline{\tilde{v}}_1| >> |\underline{\tilde{v}}_2| >> |\underline{\tilde{v}}_3| >> \dots \quad .$$

The quantities are presumed to depend on a parameter $\epsilon < 1$, so that the series will converge. The method of successive approximation is a means of finding the coefficients of the powers of ϵ.

Physically, this is a justifiable method, since we expect that the beam quantities will have a first-, second-, and third-harmonic part and that the amplitudes of these will decrease with increase in the number of the harmonic. The method is also consistent with Maxwell's equations since these are linear and hence permit superposition of solutions.

Substituting Equation (23) into Equation (17) gives:

$$\frac{dv_{rn}}{dt} + 2k_m\omega_L v_{\theta n} + \sum_{k=1}^{n-1}\left(v_{rk}\frac{\partial v_{r,n-k}}{\partial r} + v_{zk}\frac{\partial v_{r,n-k}}{\partial z} - \frac{v_{\theta K}v_{\theta,n-k}}{r}\right)$$

$$= \eta(E_{rn} + r\omega_o B_{zn} - v_{zo}B_{\theta n}) \quad , \tag{24a}$$

$$\frac{dv_{\theta n}}{dt} - 2k_m\omega_L v_{rn} + \sum_{k=1}^{n-1}\left(v_{rk}\frac{\partial v_{\theta,n-k}}{\partial r} + v_{zk}\frac{\partial v_{\theta,n-k}}{\partial z} + \frac{v_{rK}v_{\theta,n-k}}{r}\right)$$

$$= \eta(E_{\theta n} + v_{zo}B_{rn}) \quad , \tag{24b}$$

$$\frac{dv_{zn}}{dt} + \sum_{k=1}^{n-1}\left(v_{rk}\frac{\partial v_{z,n-k}}{\partial r} + v_{zk}\frac{\partial v_{z,n-k}}{\partial z}\right) = \eta(E_{zn} - r\omega_o B_{rn}) \quad . \tag{24c}$$

The terms arising from $\underline{v} \times \underline{B}$ have been ignored on the right-hand side of Equation (24). This assumption is justifiable for slow beams.

From Equations (22) we find that

$$v_{z1} = \frac{dP_{z1}}{dt} \quad , \tag{25a}$$

$$v_{\theta 1} = \frac{dP_{\theta 1}}{dt} \quad , \tag{25b}$$

$$v_{r1} = \frac{dP_{r1}}{dt} \quad , \tag{25c}$$

with

$$\underline{P} = \underline{P}_1 + \underline{P}_2 + \underline{P}_3 + \cdots \quad . \tag{26}$$

as before. We may now combine Equations (24) and (25) to find

$$\frac{d^2 P_{r1}}{dt^2} + 2k_m \omega_L \frac{dP_{\theta 1}}{dt} = \eta (E_{r1} + r\omega_o B_{z1} - v_{zo} B_{\theta 1}) \quad , \tag{27a}$$

$$\frac{d^2 P_{\theta 1}}{dt^2} - 2k_m \omega_L \frac{dP_{r1}}{dt} = \eta (E_{\theta 1} + v_{zo} B_{r1}) \quad , \tag{27b}$$

$$\frac{d^2 P_{z1}}{dt^2} = \eta (E_{z1} - r\omega_o B_{r1}) \quad . \tag{27c}$$

We notice that Equations (27) are much simplified if $k_m = 0$. As stated previously, this is the case of Brillouin flow. We may, of course, combine Equation (27) with the Maxwell equations and solve for the case $k_m \neq 0$, but the results are complicated and not easily manipulated to obtain the nonlinear solution desired (see Appendix C).

The case of Brillouin flow is interesting from another standpoint. The experimental work of Gilmour[21] shows that the assumptions at the beginning of this section are nearly satisfied for a well-designed Brillouin beam. Therefore we restrict our attention to the Brillouin case. Another reason that the case of Brillouin flow is of interest is that many practical devices are designed so that very little or no magnetic flux will thread the cathode, precisely one of the main conditions required for Brillouin flow. Although Brillouin flow is an ideal state of behavior not actually found in real devices, it is closely approached in many microwave generators.

Appendix D shows that the results of the analysis for the case of the Brillouin beam may be applied when the magnetic focusing field slightly exceeds the Brillouin value.

We assume that $B_{r1} = E_{\theta 1} = B_{z1} = 0$. In other words we see if a purely transverse magnetic solution can be obtained for the first-order equations. Combining Equation (27) with Equations (8c), (9a) and (9c), we obtain

$$\left(\frac{d^2}{dt^2} + \omega_p^2\right)\left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial P_{z1}}{\partial r} + \frac{\partial^2}{\partial z^2} P_{z1} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} P_{z1} - \frac{\omega_p^2}{c^2} P_{z1}\right) = 0 \quad . \tag{28}$$

This is the wave equation for the first-order solution. There are two possible solutions for Equation (28):

Case I:

$$P_{z1} = J_o(\gamma r)\, e^{j(\omega t - \beta_z)} \quad ,$$

where

$$\gamma^2 = -\beta^2 + \frac{\omega^2}{c^2} - \frac{\omega_p^2}{c^2} \quad ;$$

Case II:

$$P_{z1} = I_o(\gamma r)\, e^{j(\omega t - \beta_z)} \quad ,$$

where

$$\gamma^2 = \beta^2 - \frac{\omega^2}{c^2} + \frac{\omega_p^2}{c^2} \quad .$$

For each case $\nabla \cdot \underline{E}_1 = 0$. This means that the transverse electric and transverse magnetic modes are independent in the first-order solution.

Other solutions of form $P_{z1} = f(r)\, e^{j\omega_p t}$ are possible. We shall, however, find that these are not excited.

We now wish to consider the relations between the fields at the edge of the beam (but inside it), and the fields on the outside of the beam. As discussed in the introduction, the electromagnetic fields must be continuous at the edge of the beam, which is the basis for the matching of fields on the boundary. The continuity of the longitudinal electric fields at $r = b$ implies that the variations with z of the longitudinal fields inside and outside the beam will be the same. This means that if E_z is known inside the beam, it is also known outside the beam, and using Maxwell's equations one can also find E_r outside the beam.

For the methods used here for satisfying the boundary conditions, the beam must be replaced by an equivalent smooth beam with a surface charge density; otherwise the boundary conditions are dependent on the time.

The surface charge density is given by

$$\rho_s(z,t) = \int_b^{b+\Delta r} [\rho_o + \rho_1(r,z,t)]\, dr \quad ,$$

where b is the unperturbed radius of the beam, and

$$\Delta r = P_{r1}(b + \Delta r, z, t) \quad .$$

In our case $\rho_1 = 0$, and so

$$\rho_s = \rho_o \Delta r \quad . \tag{29}$$

To find Δr we expand $P_{r1}(b + \Delta r, z, t)$ about b by Taylor's theorem and drop terms of greater than first order in Δr. We have

$$\Delta r = P_{r1}(b, z, t) + \Delta r \left[\frac{\partial}{\partial r} P_{r1}(r, z, t) \right]_{r=b}$$

and

$$\Delta r \doteq P_{r1}(b, z, t) + P_{r1}(b, z, t) \left[\frac{\partial}{\partial r} P_{r1}(r, z, t) \right]_{r=b} \quad .$$

The last term is clearly a second-order quantity, so

$$\Delta r \doteq P_{r1}(b, z, t) \quad . \tag{30}$$

Combining Equations (29) and (30), we have

$$\rho_s = \rho_o P_{r1}(b, z, t) \quad . \tag{31}$$

The boundary condition may now be written as

$$\left(\frac{E_r}{E_z} \right)_{out} = \left(\frac{E_r + \frac{\rho_s}{\epsilon_o}}{E_z} \right)_{in} \quad \text{at} \quad r = b \tag{32}$$

Both E_r and E_z outside the beam are found by straightforward solution of the Maxwell equations for free space, subject to the boundary condition that E_z must be zero at the drift tube wall and the drift tube radius is a. We have two cases:

Case A:

$$E_{z\,out} = [N_o(\Gamma a)\, J_o(\Gamma r) - N_o(\Gamma r)\, J_o(\Gamma a)]\, e^{j(\omega t - \beta z)}$$

where

$$\Gamma^2 = \frac{\omega^2}{c^2} - \beta^2$$

Case B:

$$E_{z\,out} = \left[K_o(\Gamma a)\, I_o(\Gamma r) - I_o(\Gamma a)\, K_o(\Gamma r)\right] e^{j(\omega t - \beta z)}$$

where

$$\Gamma^2 = \beta^2 - \frac{\omega^2}{c^2}$$

We now combine these results with Equation (32). There are four cases for study, corresponding to combinations of the two sets of solutions inside and outside of the beam. We may tabulate these as shown in Table I.

Table I. Forms of Equation (32).

Fields Inside	Fields Outside		
	Case A	Case B	
Case I	I A	I B	Forms of Equation (32) for study
Case II	II A	II B	

Case I A in the table corresponds to the wave-guide modes. These are the electromagnetic waves that would propagate if the tunnel were considered as a wave guide partially filled with the beam. The phase velocity of the Case I A waves is approximately equal to the velocity of light. The propagation characteristics of the waves are slightly different from the characteristics of waves propagating in a vacuum-filled guide, but this is due, of course, to the presence of the beam.

Case I B has no solution under any conditions. There are thus no modes of propagation corresponding to Case I B. Case II A has solutions of form $e^{j(\omega t)}\, e^{-\alpha z}$ only. These are exponentially damped electromagnetic waves and correspond to wave-guide modes below cut-off frequency.

None of these cases is of interest. For case II B, however, Equation (32) may be written as:

$$\frac{1}{\gamma b}\left(1 - \frac{\omega_p^2}{\omega_q^2}\right)\frac{I_1(\gamma b)}{I_o(\gamma b)} = \frac{1}{b}\,\frac{[K_o(\Gamma a)\, I_1(\Gamma b) + I_o(\Gamma a)\, K_1(\Gamma b)]}{[K_o(\Gamma a)\, I_o(\Gamma b) - I_o(\Gamma a)\, K_o(\Gamma b)]} \quad . \tag{33}$$

To derive Equation (33) we define

$$\beta = \frac{\omega}{u_o} \pm \frac{\omega_q}{u_o} \quad ,$$

or

$$\beta = \beta_e \pm \beta_{q1} \quad .$$

From the form of Equation (33), we note that $\omega_q < \omega_p$. Usually $\omega_p \ll \omega$ and we may let $\omega_q \ll \omega$; therefore, since

$$\Gamma^2 = \beta^2 - \frac{\omega^2}{c^2} \quad ,$$

and

$$\gamma^2 = \beta^2 - \frac{\omega^2}{c^2} + \frac{\omega_p^2}{c^2} \quad ,$$

we may set $\Gamma \doteq \gamma \doteq \beta_e$. Define $A = \beta_e a$, $B = \beta_e b$, and $R = \beta_e r$. Then we can write Equation (33) as

$$R_1^2 = B I_1(B)\left[K_o(B) - \frac{I_o(B)\, K_o(A)}{I_o(A)}\right] \quad , \tag{34}$$

where

$$R_1^2 = \frac{\omega_q^2}{\omega_p^2} \quad .$$

Figure 4 plots R_1^2 for various values of B/A.

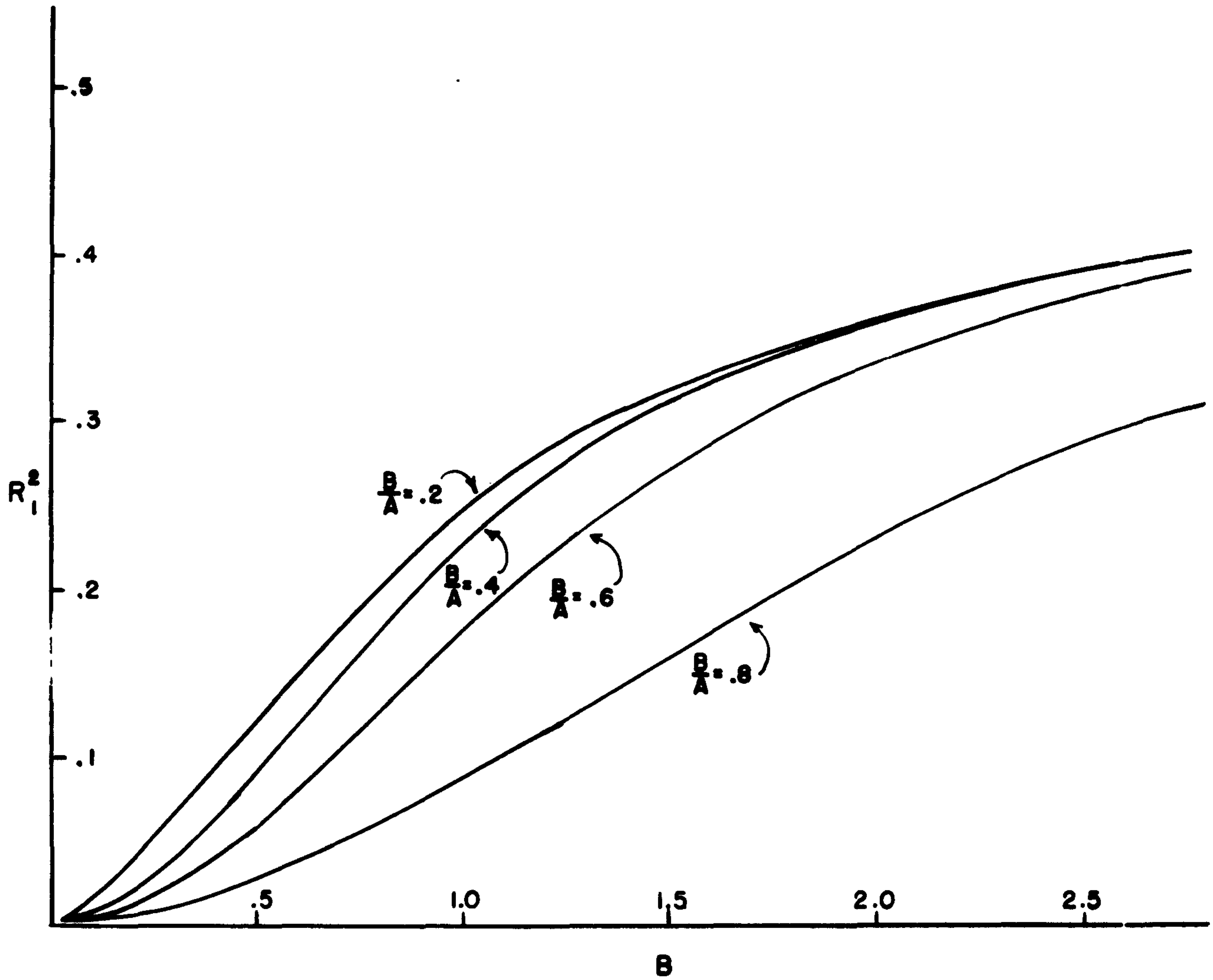

Figure 4. Values of Reduction Constant R_1^2 for Various Beam Geometries.

The quantity ω_q is the reduced plasma frequency. In any finite structure, oscillations of the electrons will not occur at the plasma frequency, since the fringing of electric fields to the wall will reduce the electric field acting on an electron. Therefore $\omega_q < \omega_p$.

II. MODULATION BY A GRIDLESS GAP

We now seek to find a set of suitable initial conditions that could be used to specify the modulation at $z = 0$. Since the modulation is presumed to be strong, we ignore the space-charge effects in the modulating region. Let E_{cr} and E_{cz} be the components of the modulating field, and define the following quantities:

$$\xi_r = \frac{v_r}{v_{zo}} \qquad \xi_z = \frac{v_z}{v_{zo}} \qquad \xi_\theta = \frac{v_\theta}{v_{zo}} \quad ,$$

$$T = \omega t \qquad R = \beta_e r \qquad Z = \beta_e z \quad .$$

Now let

$$a = \frac{V_1}{V_o} \tag{35}$$

be the modulation parameter; V_1 is the potential across the gap and V_o is the kinetic voltage of electrons entering the gap. A simple energy calculation gives:

$$V_c = \frac{v_{zo}^2}{2\eta} \quad .$$

Let

$$e_{c_r} = \frac{E_{cr}}{\beta_e V_o a} \quad , \qquad e_{c_z} = \frac{E_{cz}}{\beta_e V_o a} \quad ,$$

and

$$H = 2\frac{\eta B_o}{\beta_e v_{zo}} = \frac{4\omega_L}{\omega} \quad .$$

Let us further assume

$$\xi_r = \sum_{n=1}^{\infty} \xi_{rn}\, a^n \quad ,$$

$$\xi_\theta = \frac{R\omega_o}{\omega} \sum_{n=1}^{\infty} \xi_{\theta n}\, a^n \quad ,$$

$$\xi_z = 1 + \sum_{n=1}^{\infty} \xi_{zn}\, a^n \quad .$$

It may be seen that this is merely a successive approximation with a as the series parameter ϵ.

Now putting the foregoing quantities into Equation (5) and equating terms in like powers of a gives

$$\frac{\partial \xi_{r1}}{\partial T} + \frac{\partial \xi_{r1}}{\partial Z} = \frac{1}{2}\,\xi_{cr} + \frac{1}{2}\,\xi_{\theta 1}\left(\frac{4\omega_o}{\omega} + H\right) \quad , \tag{36a}$$

$$\frac{\partial \xi_{\theta 1}}{\partial T} + \frac{\partial \xi_{\theta 1}}{\partial Z} = \frac{1}{2}\,\xi_{r1}\left(\frac{4\omega_o}{\omega} + H\right) \quad , \tag{36b}$$

$$\frac{\partial \xi_{z1}}{\partial T} + \frac{\partial \xi_{rn}}{\partial Z} = \frac{1}{2}\, e_{cz} \quad , \tag{36c}$$

and for $n = 2, 3, \ldots,$

$$\frac{\partial \xi_{rn}}{\partial T} + \frac{\partial \xi_{rn}}{\partial Z} = -\left[\sum_{k=1}^{n-1}\left(\xi_{r,n-k}\frac{\partial \xi_{rk}}{\partial R} + \xi_{z,n-k}\frac{\partial \xi_{rk}}{\partial Z} - \frac{\xi_{\theta,n-k}\,\xi_{\theta k}}{R}\right)\right] + \frac{1}{2}\,\xi_{\theta n}\left(\frac{4\omega_o}{\omega} + H\right) \quad , \tag{36d}$$

$$\frac{\partial \xi_{\theta n}}{\partial T} + \frac{\partial \xi_{\theta n}}{\partial Z} = -\left[\sum_{k=1}^{n-1}\left(\xi_{r,n-k}\frac{\partial \xi_{\theta k}}{\partial R} + \xi_{z,n-k}\frac{\partial \xi_{\theta k}}{\partial Z} + \frac{\xi_{r,n-k}\,\xi_{\theta k}}{R}\right)\right] - \frac{1}{2}\xi_{rn}\left(\frac{4\omega_o}{\omega} + H\right), \tag{36e}$$

$$\frac{\partial \xi_{zn}}{\partial T} + \frac{\partial \xi_{zn}}{\partial Z} = -\left[\sum_{k=1}^{n-1}\left(\xi_{r,n-k}\frac{\partial \xi_{zk}}{\partial R} + \xi_{z,n-k}\frac{\partial \xi_{zk}}{\partial Z}\right)\right] . \tag{36f}$$

Equations (36) represent a normalized form of Equation (17) suitable for ballistic studies of motions in gaps. For the Brillouin case $\frac{4\omega_o}{\omega} + H = 0$ and the equations assume the simple form,

$$\frac{\partial \xi_{r1}}{\partial T} + \frac{\partial \xi_{r1}}{\partial Z} = \frac{1}{2}\, e_{cr} \quad , \tag{37a}$$

$$\frac{\partial \xi_{z1}}{\partial T} + \frac{\partial \xi_{z1}}{\partial Z} = \frac{1}{2}\, e_{cz} \quad , \tag{37b}$$

and for $n = 2, 3, 4, \ldots,$

$$\frac{\partial \xi_{rn}}{\partial T} + \frac{\partial \xi_{rn}}{\partial Z} = -\left[\sum_{k=1}^{n-1}\left(\xi_{r,n-k}\frac{\partial \xi_{rk}}{\partial R} + \xi_{z,n-k}\frac{\partial \xi_{zk}}{\partial Z}\right)\right] , \tag{37c}$$

$$\frac{\partial \xi_{zn}}{\partial T} + \frac{\partial \xi_{zn}}{\partial Z} = -\left[\sum_{k=1}^{n-1}\left(\xi_{r,n-k}\frac{\partial \xi_{zk}}{\partial R} + \xi_{z,n-k}\frac{\partial \xi_{zk}}{\partial Z}\right)\right] , \tag{37d}$$

$$\xi_{\theta 1} = \xi_{\theta n} = 0 \quad . \tag{37e}$$

It should be noted that only the solutions to Equations (36a-c) and (37a and b) that result from the driving functions e_{cr} and e_{cz} may be allowed; that is, no complementary solutions may be admitted. The reason for this is that only the motions of the electrons actually produced by the modulating fields may exist in a real beam, and allowing complementary solutions would permit us to say that motions exist that arise spontaneously. Similarly, no complementary solutions may be allowed in any of the other Equations (36) and (37).

The fields E_{cr} and E_{cz} in the gap have been found by Wang[22] but they are too complicated for convenient use in Equations (37a) and (37b). We may, however, use a simpler method to arrive at initial conditions. In the usual case,

$$e_{cr} = 2G_r(R, Z)\ e^{jT} \quad ,$$

$$e_{cz} = F_z(R, Z)\ e^{jT} \quad .$$

Since the electric fields arising from electron motions in the gap are ignored, we can assume that only electric fields of frequency ω are present. If

$$\xi_{r1} = F_r(R, Z)\ e^{jT} \quad ,$$

$$\xi_{z1} = F_z(R, Z)\ e^{jT} \quad ;$$

then

$$\frac{\partial F_r}{\partial Z} + jF_r = G_r \quad ,$$

$$\frac{\partial F_z}{\partial Z} + jF_z = G_z \quad ,$$

and

$$\xi_{r1} = e^{j(T-Z)} \int G_r \, e^{jZ} \, dZ \quad ,$$

$$\xi_{z1} = e^{j(T-Z)} \int G_z \, e^{jZ} \, dZ \quad .$$

Further suppose that

$$F_r = f_r(R, Z) \; e^{-jZ}$$

$$F_z = f_z(R, Z) \; e^{-jZ} \quad ;$$

then

$$f_r(R, \infty) - f_r(R, -\infty) = \int_{-\infty}^{\infty} G_r \, e^{jZ} \, dZ$$

$$f_z(R, \infty) - f_z(R, -\infty) = \int_{-\infty}^{\infty} G_z \, e^{jZ} \, dZ \quad .$$

Now $f(R, \infty)$ is the value at the exit from the modulation region, and since ξ_{r1} and ξ_{z1} must be zero at the entrance to the gap, $f(R, -\infty) = 0$. After modulation,

$$\xi_{z1_{exit}} = e^{j(T-Z)} \int_{-\infty}^{\infty} G_z \, e^{jZ} \, dZ \quad , \tag{38a}$$

$$\xi_{r1_{exit}} = e^{j(T-Z)} \int_{-\infty}^{\infty} G_r \, e^{jZ} \, dZ \quad . \tag{38b}$$

We may identify the Fourier transforms with the gap-coupling coefficients. If we apply the Maxwell equations to find G_z, we have

$$\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial G_z}{\partial R}\right)+\frac{\partial^2 G_z}{\partial Z^2}+\frac{v_{zo}^2}{c^2}G_z = 0 \quad .$$

Since $v_{zo}^2 \ll c^2$, this becomes

$$\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial G_z}{\partial R}\right)+\frac{\partial^2 G_z}{\partial Z^2} = 0 \quad ,$$

from which

$$G_z = J_o(KR)\ e^{-jKZ} \quad .$$

where $-K^2$ is the separation constant. The gap fields are given by

$$G_z = \int_{-\infty}^{\infty} A(K)\ J_o(KR)\ e^{-jKZ}\ dK \quad ,$$

where $A(K)$ is chosen so that G_z satisfies the boundary conditions. Substituting this in Equation (38a) gives

$$\xi_{z1_{exit}} = 2\pi A(1)\ I_o(R)\ e^{j(T-Z)} \quad .$$

At the edge of the gap

$$E_z = \frac{\alpha v_o}{d}\ e^{jT} \quad ,$$

if d is the gap width; then

$$G_z = \frac{1}{2D}$$

where $D = \beta_e d$ and

$$g_z = \int_{-D/2}^{D/2} \frac{1}{2D} e^{jZ} dZ = \frac{\sin \frac{D}{2}}{D} \quad .$$

$$\xi_{z1_{exit}} = \frac{\sin \frac{D}{2}}{D} \frac{I_o(R)}{I_o(A)} e^{j(T-Z)} \quad . \tag{39}$$

In a similar manner, we find

$$\xi_{r1_{exit}} = j \frac{\sin \frac{D}{2}}{D} \frac{I_1(R)}{I_o(A)} e^{j(T-Z)} \quad . \tag{40}$$

We also require that the polarization distances be zero at exit from the gap. The polarization distances are zero at entrance to the gap, and if the gap is short, we may expect that there will not be an appreciable displacement in the gap. These are the initial conditions for the first-order solution. We must now establish conditions for the higher-order solutions.

Of course, the requirement that the polarizations be zero will not be changed. In addition, the a-c charge density is zero at the entrance to the gap. Since $\nabla \quad \underline{E}_c = 0$, there can be no density modulation in the gap. Thus we must have $\rho = 0$ at the exit from the gap. Since there are presumed to be no electric fields in the gap with frequency 2ω or higher, we expect that no solutions of form,

$$I_o(2R) \; e^{2j(\omega t - \beta z)} \quad ,$$

will be excited. The origin for the z co-ordinate is at the center of the gap as already indicated. The actual point at which the solutions to the wave equation are matched to the initial conditions, however, is at the point where the electrons have just left the influence of the fields of the modulating gap. In practical cases this should not make any difference, as the length of the gap will usually be short compared to the reduced plasma wavelength.

For convenience we define

$$\phi = T - Z = \omega t - \beta_e z \quad . \tag{41}$$

The two waves are possible as first-order solutions: a fast wave,

$$P_{z1} = A_1 I_o(R) \, e^{j\beta_q z} \, e^{j\phi} \quad ,$$

and a slow wave,

$$P_{z1} = A_2 I_o(R) \, e^{-j\beta_q z} \, e^{j\phi} \quad .$$

If we follow the assumption made above, $\beta_q \ll \beta_e$, we may add these two waves to produce the following solutions which match the initial conditions at $z = 0$:

$$P_{z1} = M \sin\beta_q z \, I_o(R) \cos\phi \quad . \tag{42a}$$

$$P_{\gamma 1} = -M \sin\beta_q z \, I_1(R) \sin\phi \tag{42b}$$

$$P_{\theta 1} = v_{\theta 1} = \rho_1 = 0 \quad , \tag{42c}$$

$$v_{z1} = M\omega_q I_o(R) \cos\beta_q z \cos\phi \quad , \tag{42d}$$

$$v_{\gamma 1} = M\omega_q I_1(R) \cos\beta_q z \sin\phi \quad , \tag{42e}$$

where

$$M = \frac{a \sin \frac{D}{2}}{2D\beta_q I_o(A)} \quad . \tag{42f}$$

Similar results have been described by Chodorow and Zitelli.[23]

We shall now use these results to obtain the second-order solution. From Equations (24):

$$\frac{d}{dt}(v_{r2} + Q_{r2}) = \eta(E_{r2} + \omega_o B_{z2} - v_{zo} B_{\theta 2}) \quad , \tag{43a}$$

$$\frac{d}{dt}(v_{z2} + Q_{z2}) = \eta(E_{z2} - r\omega_o B_{r2}) \quad , \tag{43b}$$

where

$$\frac{dQ_{r2}}{dt} = v_{r1} \frac{\partial v_{r1}}{\partial r} + v_{z1} \frac{\partial v_{r1}}{\partial z} \quad , \tag{43c}$$

$$\frac{dQ_{z2}}{dt} = v_{r1} \frac{\partial v_{z1}}{\partial r} + v_{z1} \frac{\partial v_{z1}}{\partial z} \quad . \tag{43d}$$

The terms $\underline{v}_1 \times \underline{B}_1$ are negligible, as assumed. As before we shall assume that $B_{z2} = B_{r2} = 0$. We now combine Equations (43) with Equations (8c), (9a), and (9c). Two equations result:

$$\left(\frac{\omega_p^2}{c^2} v_{zo} - \frac{\partial}{\partial z}\frac{d}{dt}\right)\frac{\partial E_{z2}}{\partial r} = \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\omega_p^2}{c^2} - \frac{\partial^2}{\partial z^2}\right)\frac{dE_{r2}}{dt} - \frac{\omega_p^2}{\eta c^2}\frac{\partial}{\partial t}\frac{dQ_{r2}}{dt} \quad ,$$

and

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r}\frac{dE_{z2}}{dt} - \frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{d^2}{dt^2} + \frac{\omega_p^2}{c^2}\right)E_{z2}$$

$$= \left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial z} + \frac{v_{zo}}{c^2}\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial t}\right)\frac{dE_{r2}}{dt} - \frac{\omega_p^2}{\eta c^2}\frac{\partial}{\partial t}\frac{dQ_{z2}}{dt} \quad .$$

These may be combined to yield

$$\left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\omega_p^2}{c^2}\right)\left(\frac{d^2}{dt^2} + \omega_p^2\right)E_{z2}$$

$$= \frac{\omega_p^2}{\eta}\left[\frac{1}{r}\frac{\partial}{\partial r} r \left(\frac{\partial}{\partial z} + \frac{v_{zo}}{c^2}\frac{\partial}{\partial t}\right)\frac{dQ_{r2}}{dt} + \left(\frac{\partial^2}{\partial_z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\omega_p^2}{c^2}\right)\frac{dQ_{z2}}{dt}\right] .$$

This equation has the same form as Equation (28), which was used to obtain the first-order solution, except that the quantities on the right-hand side are drive terms. If we take advantage of our assumptions, $u_o^2 \ll c^2$ and $\omega_p^2 \ll \omega^2$, we can reduce the foregoing equation to the form,

$$\left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{d^2}{dt^2} + \omega_p^2\right) E_{z2} = \frac{\omega_p^2}{\eta}\left[\frac{1}{r}\, r \frac{\partial}{\partial z}\frac{dQ_{r2}}{dt} + \frac{\partial^2}{dz^2}\frac{dQ_{z2}}{dt}\right] . \tag{44}$$

This is the wave equation for the second-order solution analogous to the Equation (28) in the first-order case.

The solution of Equation (44) is straightforward. If the approximation, $\beta_q \ll \beta_e$, is applied,

$$\frac{\partial v_{z1}}{\partial r} = \frac{\partial v_{r1}}{\partial z} \quad ,$$

and from this fact it follows that

$$\frac{\partial}{\partial z}\frac{dQ_{r2}}{dt} = \frac{\partial}{\partial r}\frac{dQ_{z2}}{dt} \quad .$$

Thus Equation (44) becomes

$$\left(\frac{1}{\omega_p^2}\frac{d^2}{dt^2} + 1\right)\eta E_{z2} = \frac{dQ_{z2}}{dt} \quad . \tag{45}$$

In a similar manner we find

$$\frac{1}{\omega_p^2}\frac{d^2}{dt^2} + 1\eta E_{r2} = \frac{dQ_{r2}}{dt} \quad . \tag{46}$$

Now dQ_{z2}/dt and dQ_{r2}/dt are calculated from Equations (42) and (43) and v_{r2} and v_{z2} are found from Equations (43), (45), and (46), giving

$$v_{z2} = \frac{M^2\beta_e\omega_q}{2}\left\{\left[I_o^2(R) - I_1^2(R)\right]\left[\frac{R_1^2}{1-4R_1^2}\sin 2\beta_q z - \frac{R_1 - 2R_1^3}{1-4R_1^2}\sin\beta_p q\right]\sin 2\phi\right\}, \tag{47a}$$

$$v_{r2} = \frac{M^2\beta_e\omega_q}{2}\left\{\left[\frac{R_1^2}{1-4R_1^2}\sin 2\beta_q z - \frac{R_1 - 2R_1^3}{1-4R_1^2}\sin\beta_p\right]\right.$$

$$\left.\left[\frac{I_1^2(R)}{R}\cos 2\phi + 2I_o(R)\,I_1(R) - \frac{I_1^2(R)}{R}\right]\right\} \quad . \tag{47b}$$

The terms in $\sin\beta_p z$ are complementary solutions added to insure that $\tilde{\rho}_2 = 0$ at $z = 0$, as required by the conditions of modulation. The charge density $\tilde{\rho}_2$ is given by

$$\tilde{\rho}_2 = \rho_o \frac{M^2\beta_e^2 R_1^2}{2}\left\{\left[1 + \frac{\cos 2\beta_q z}{1-4R_1^2} - \frac{2-4R_1^2}{1-4R_1^2}\cos\beta_p z\right]\right.$$

$$\left[\left(\frac{I_1(R)\,I_o(R)}{R} - \frac{I_1^2(R)}{R^2} - I_o^2(R) + I_1^2(R)\right)\cos 2\phi\right.$$

$$\left.\left. + \frac{I_1^2(R)}{R^2} - \frac{I_1(R)\,I_o(R)}{R} + I_1^2(R) + I_1^2(R) + I_o^2(R)\right]\right\} \quad . \tag{47c}$$

Using Equations (47c) and (9b), we may formulate an equation for E_θ and deduce v_θ. It is found that v_θ is a function of R and z similar to Equations (47a) and (47b), but multiplied by the coefficient,

$$\frac{M^2\beta_e\omega_q}{2}\left(\frac{2}{R_1^2}\ \frac{\omega_L}{\omega}\ \frac{v_{zo}^2}{c^2}\right) \quad .$$

In any usual case, $\frac{2}{R_1^2}\ \frac{\omega_L}{\omega}\ \frac{v_{zo}^2}{c^2}$ will be so much less than one, that for all practical purposes we have

$$v_{\theta 2} = E_{\theta 2} = B_{r2} = b_{z2} \doteq 0 \quad . \tag{47d}$$

This justifies the assumption made that $B_{z2} = B_{r2} = 0$.

We now seek to find the polarization distances P_{r2} and P_{z2}. Expanding Equations (22a) amd (22b), we find

$$v_{r2} = \frac{dP_{r2}}{dt} + \frac{\partial P_{r1}}{\partial r}\ \frac{dP_{r1}}{dt} + \frac{\partial P_{r1}}{\partial z}\ \frac{dP_{z1}}{dt} \quad ,$$

$$v_{z2} = \frac{dP_{z2}}{dt} + \frac{\partial P_{z1}}{\partial r}\ \frac{dP_{r1}}{dt} + \frac{\partial P_{z1}}{\partial z}\ \frac{dP_{z1}}{dt} \quad .$$

Combining these with Equations (47a and b), we have:

$$P_{z2} = -\frac{M^2\beta_e R_1^2}{4}\left\{\left[\frac{1+2R_1^2}{2R_1^2} + \left(\frac{1}{1-4R_1^2} - \frac{1}{2R_1^2}\right)\cos 2\beta_q z - \frac{2-4R_1^2}{1-4R_1^2}\cos\beta_p z\right]\left[(I_o^2(R) - I_1^2(R)\ \sin 2\phi\right]\right\} \quad , \tag{47e}$$

and

$$P_{r2} = -\frac{M^2\beta_e R_1^2}{4}\left\{\left[\frac{1+2R_1^2}{2R_1^2} + \left(\frac{1}{1-4R_1^2} - \frac{1}{2R_1^2}\right)\cos 2\beta_q z - \frac{2-4R_1^2}{1-4R_1^2}\cos\beta_p z\right]\left[\frac{I_1^2(R)}{R}\cos 2\phi + \left(2I_o(R)\,I_1(R) - \frac{I_1^2(R)}{R}\right)\right]\right\} . \tag{47f}$$

We note that the equilibrium radius of the beam is shifted after the excitation occurs, since P_{r2} contains a part which does not vary with time. Figure 5 shows the effect of a disturbance on the outer radius of a beam.

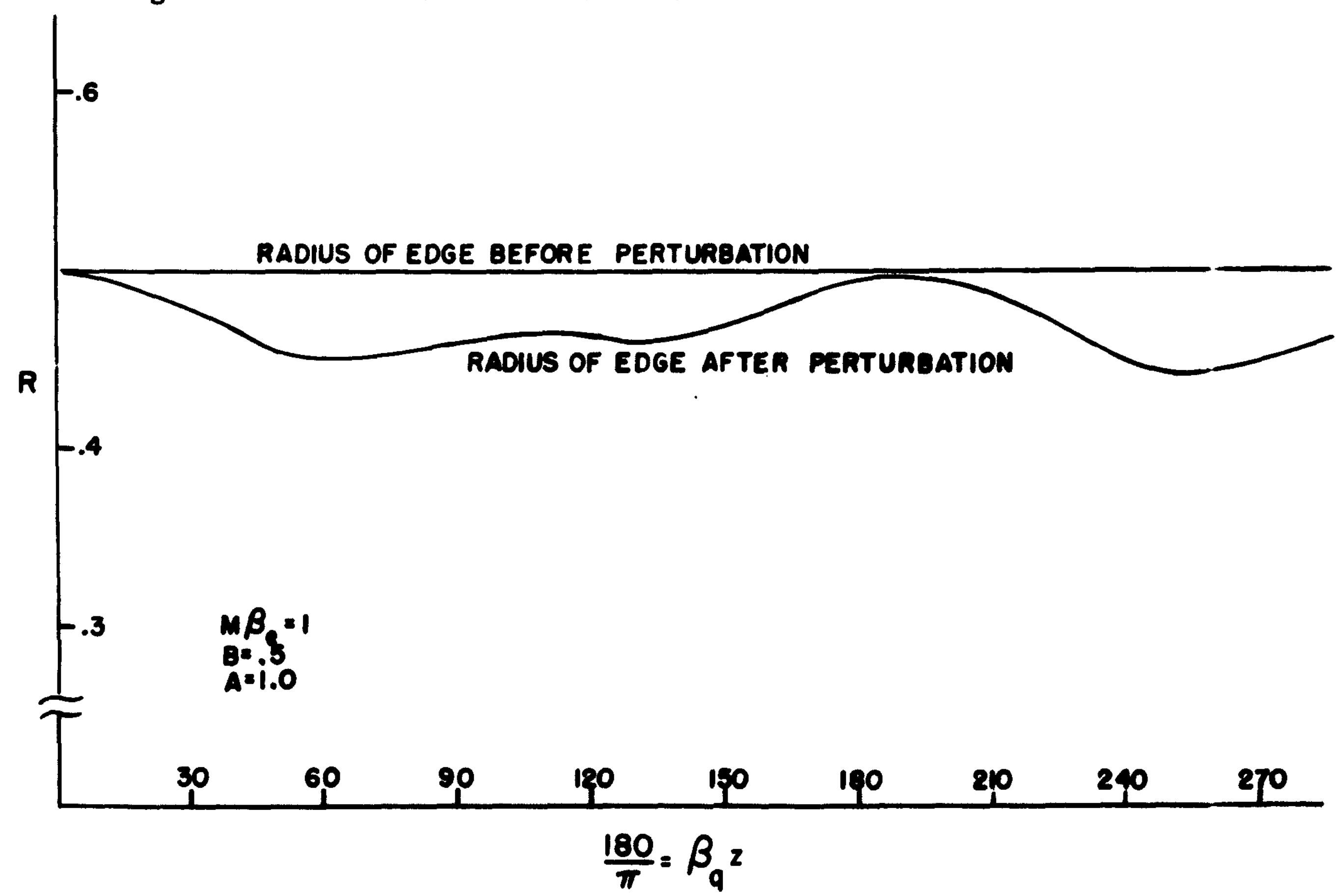

Figure 5. Distortion of "D-C" Edge of Beam by R-F Disturbance.

This fact will make it difficult to develop a method of calculating the surface charge density as was used in the first-order solution. However, if we assume that E_{z2} must be continuous at $r = b$, we may find E_{r2} and E_{z2} outside the beam. Then ρ_{s2}/ϵ_o will be equal to the difference between the values of E_{r2} inside and outside the beam at $r = B$. Following this, we find

$$\rho_{s2} = C_{B2}\left[1 + \frac{\cos 2\beta_q z}{1 - 4R_1^2} - \frac{2 - 4R_1^2}{1 - 4R_1^2}\cos\beta_p z\right]\cos 2\phi \quad , \tag{48a}$$

where

$$C_{s2} = \frac{\rho_o M^2 \beta_e R_1^2}{4}\left[\frac{I_o^2(B) - I_1^2(B)}{K_o(2A)\, I_o(2B) - K_o(2B)\, I_o(2A)} - \frac{I_1^2(B)}{B}\right] . \tag{48b}$$

Only the a-c fields are used in computing the surface charge, since the d-c parts merely distort the steady-state solution and do not produce a-c fields in the region outside the beam. This completes the second-order solution. We now consider the third-order case.

The third-order solution may be obtained in much the same way as the second-order solution. We may write

$$\frac{d}{dt}\,(v_{r3} + Q_{r3}) = \eta(E_{r3} - v_{zo} B_{\theta 3}) \quad , \tag{49a}$$

$$\frac{d}{dt}\,(v_{z3} + Q_{z3}) = \eta E_{z3} \quad , \tag{49b}$$

where

$$\frac{dQ_{r3}}{dt} = v_{r1}\frac{\partial v_{r2}}{\partial r} + v_{r2}\frac{\partial v_{r1}}{\partial r} + v_{z1}\frac{\partial v_{r2}}{\partial z} + v_{z2}\frac{\partial v_{r1}}{\partial z} \quad , \tag{49c}$$

$$\frac{dQ_{z3}}{dt} = v_{r1} \frac{\partial v_{z2}}{\partial r} + \frac{\partial v_{z1}}{\partial r} + v_{z1} \frac{\partial v_{z2}}{\partial z} + v_{z2} \frac{\partial v_{z1}}{\partial z} \quad . \tag{49d}$$

If we apply the same approximations as made in the second-order solution and combine Equations (49) with Equations (8c), (9a and c) we find

$$\left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{d^2 E_{z3}}{dt^2} + \omega_p^2 E_{z3}\right)$$

$$= \frac{\omega_p^2}{\eta}\left[\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial z}\left(\frac{dQ_{r3}}{dt} - \frac{d}{dt}\frac{\tilde{\rho}_2}{\rho_o} v_{r1}\right) + \frac{\partial^2}{\partial z^2}\left(\frac{dQ_{z3}}{dt} - \frac{d}{dt}\frac{\rho_2}{\rho_o} v_1\right)\right] . \tag{50}$$

This is the wave equation for the third-order solution. The terms involving ρ_2 arise from the current density in the Maxwellian equation,

$$\frac{1}{\mu_o} \nabla \times \underline{B} = \underline{J} + \epsilon_o \frac{\partial \underline{\underline{E}}}{\partial t} \quad .$$

The solution of Equation (50) is not as straightforward as was the solution of Equation (44) in the second-order case. The Q_{r3} and Q_{z3} have the property that

$$\frac{\partial}{\partial z}\frac{dQ_{r3}}{dt} = \frac{\partial}{\partial r}\frac{dQ_{z3}}{dt} \quad ,$$

but this is not the case for the terms involving $\tilde{\rho}_2$.

To deal with the part of the drive terms involving $\tilde{\rho}_2$ we must make some approximation. An exact solution may be found for part of these drive terms. For the remaining part we may approximate the drive terms by the value they assume at $R = 0$. Figure 6 shows the basis for this method.

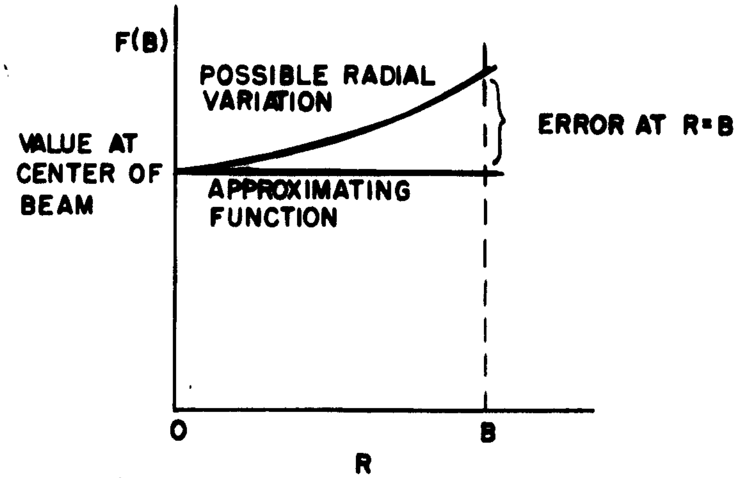

Figure 6. Method of Approximation Used in Calculating Part of Third-Order Solution.

If $B < 1$ the error should be reasonably small. Thus this approximation method assumes a relatively thin beam. The method used has the additional merit that the mathematical expressions used as drive terms in the solution of Equation (50) are more likely to be accurate at the center of the beam than at the outside, which is in the boundary layer region. For the case taken to compute the values of beam parameters shown in Figures 7-11, it was found that the error at $R = B$, as shown in Figure 6, was less than 10 per cent for all the drive terms used.

After calculating the drive terms dQ_{r3}/dt and dQ_{z3}/dt and using Equations (49) and (50), we find:

$$v_{z3}^{3\omega} = \frac{M^3\beta_e^2\omega_q}{8}\Bigg\{\left[3I_o^3(R) - 3I_1^2(R)\,I_o(R) + \frac{I_1(R)}{R}\right]\left[C_1\cos 3\beta_q z + C_2\cos\beta_q z\right.$$
$$\left. + C_3\cos(\beta_p - \beta_q)z + C_4\cos(\beta_p + \beta_q)z\right] + \left[I_o^3(R) - \tfrac{1}{4}\right]\left[C_5\cos 3\beta_q z\right.$$
$$\left. + C_6\cos\beta_q z + C_7\cos(\beta_p - \beta_q)z + C_8\cos(\beta_p + \beta_q)z\right]\Bigg\}\cos 3\phi \qquad (51a)$$

The values of the constants are listed in Appendix E.

$$v_{z3}^{\omega} = \frac{M^3\beta_e^2\rho_q}{8}\left\{\left[I_o^3(R) - 3I_1^2(R)\,I_o(R) + \frac{3I_1^3(R)}{R}\right]\left[C_1 \cos 3\beta_q z + C_2 \cos\beta_q z\right.\right.$$

$$\left. + C_3 \cos(\beta_p - \beta_q) z + C_4 \cos(\beta_p + \beta_q) z\right] + \left[I_o^3(R)\right]\left[C_5 \cos 3\beta_q z\right.$$

$$\left.\left. + C_6 \cos\beta_q z + C_7 \cos(\beta_p - \beta_q) z + C_8 (\beta_p + \beta_q) z\right]\right\} \cos\phi \qquad (51b)$$

The $-\frac{1}{2}$ in the second term of Equation (51a) is the result of the approximate solution. By similar methods, we find

$$v_{r3}^{3\omega} = -\frac{M^3\beta_e^2\omega_q}{8}\left\{\left[3I_o^2(R)\,I_1(R) - 3I_1^3(R) + \frac{12I_1^3(R)}{R^2} - \frac{3I_1^2(R)\,I_o(R)}{R}\right]\right.$$

$$\left[C_1 \cos 3\beta_q z + C_2 \cos\beta_q z + C_3 \cos(\beta_p - \beta_q) z + C_4 \cos(\beta_p + \beta_q) z\right]$$

$$+ \left[I_o^2(R)\,I_1(R) - \frac{1}{6}\right]\left[C_5 \cos 3\beta_q z + C_6 \cos\beta_q z\right.$$

$$\left.\left. + C_7 \cos(\beta_p - \beta_q) z + C_8 \cos(\beta_p + \beta_q) z\right]\right\} \sin 3\phi \quad , \qquad (51c)$$

and

$$v_{r3}^{\omega} = \frac{M^3\beta_e^2\omega_q}{8}\left\{\left[3I_o^2(R)\,I_1(R) + 3I^3(R) - \frac{15I_o(R)\,I_1^2(R)}{R}\right.\right.$$

$$\left. + \frac{12I_1^3(R)}{R^2}\right]\left[C_1 \cos 3\beta_q z + C_2 \cos\beta_q z + C_3 \cos(\beta_p - \beta_q) z\right.$$

$$\left. + C_4 \cos(\beta_p + \beta_q) z\right] + \left[3I_o^2(R)\,I_1(R)\right]\left[C_5 \cos 3\beta_q z + C_6 \cos\beta_q z\right.$$

$$\left.\left. + C_7 \cos(\beta_p - \beta_q) z + C_8 \cos(\beta_p + \beta_q) z\right]\right\} \sin\phi \quad . \qquad (51d)$$

The charge density is given by

$$\tilde{\rho}_3^{3\omega} = \rho_o \frac{M^3 \beta_e^3 R_1^2}{8} \Bigg\{ \Bigg[6I_o(R)\, I_1^2(R) - 6I_o^3(R) + 12\,\frac{I_1^3(R)}{R} - 6\,\frac{I_o^2(R)\, I_1(R)}{R}$$

$$- 48\,\frac{I_1^3(R)}{R^3} + 42\,\frac{I_1^2(R)\, I_o(R)}{R^2} \Bigg] \Big[C_9 \sin 3\beta_q z + C_{10} \sin\beta_q z + C_{11} \sin(\beta_p - \beta_q) z$$

$$+ C_{12} \sin(\beta_q + \beta_p) z \Big] + \Big[2I_o(R)\, I_1^2(R) - 2I_o^3(R) + \frac{3}{2} \Big] \Big[C_{13} \sin 3\beta_q z$$

$$+ C_{14} \sin\beta_q z + C_{15} \sin(\beta_p - \beta_q) z + C_{16} \sin(\beta_q + \beta_p) z \Big] \Bigg\} \sin 3\phi \quad , \tag{51e}$$

and

$$\tilde{\rho}_3^{\omega} = \rho_o \frac{M^3 \beta_e^3 R_1^2}{8} \Bigg\{ \Bigg[15\, I_1^3(R) + 9\,\frac{I_1^3(R)}{R} - 66\,\frac{I_1^2(R)\, I_o(R)}{R^2} + 30\,\frac{I_o^2(R)\, I_1(R)}{R}$$

$$+ 24\,\frac{I_1^3(R)}{R^3} - 4I_o^3(R) - 4I_1^2(R)\, I_o(R) \Bigg] \Big[C_9 \sin 3\beta_q z + C_{10} \sin\beta_q z$$

$$+ C_{11} \sin(\beta_p - \beta_q) z + C_{12} (\beta_q + \beta_p) z \Big] + \Big[2I_o^3(R) - 2I_o(R)\, I_1^2(R) \Big] \Big[C_{13} \sin 3\beta_q z$$

$$+ C_{14} \sin\beta_q z + C_{15} \sin(\beta_p - \beta_q) z + C_{16} \sin(\beta_p + \beta_q) z \Big] \Bigg\} \sin\phi \tag{51f}$$

We may compute the surface charge in the same manner as in the second-order solution and find:

$$\rho_{s3}^{3\omega} = \rho_o \frac{M^3 \beta_e^2 R_1^2}{8} \left\{ C_{s31} \left[C_9 \sin 3\beta_q z + C_{10} \sin\beta_q z + C_{11} \sin(\beta_p - \beta_q) z \right.\right.$$

$$\left. + C_{12} \sin(\beta_p + \beta_q) z \right] + C_{s32}^{3\omega} \left[C_{13} \sin 3\beta_q z + C_{14} \sin\beta_q z \right.$$

$$\left.\left. + C_{15} \sin(\beta_p - \beta_q) z + C_{16} \sin(\beta_p + \beta_q) z \right] \right\} \sin 3\phi \quad , \tag{51g}$$

$$\rho_{s3}^{\omega} = \rho_o \frac{M^3 \beta_e^2 R_1^2}{8} \left\{ C_{s31} \left[C_9 \sin 3\beta_q z + C_{10} \sin\beta_q z + C_{11} \sin(\beta_p - \beta_q) z \right.\right.$$

$$\left. + C_{12} \sin(\beta_p + \beta_q) z \right] + C_{s32}^{\omega} \left[C_{13} \sin 3\beta_q z + C_{14} \sin\beta_q z \right.$$

$$\left.\left. + C_{15} \sin(\beta_p - \beta_q) z + C_{16} \sin(\beta_p + \beta_q) z \right] \right\} \sin \phi \quad , \tag{51h}$$

where

$$C_{s31}^{3\omega} = 3I_o^2(B)\, I_1(B) - 3I_1^3(B) + \frac{12I_1^3(B)}{B^2} - \frac{3I_1^2(B)\, I_o(B)}{B}$$

$$- \left[3I_o^3(B) - 3I_1^2(B) \left(I_o(B) + \frac{I_1(B)}{B} \right) \right] \frac{K_o(3A)\, I_1(3B) + I_o(3A)\, K_1(3B)}{K_o(3A)\, I_o(3B) - I_o(3A)\, K_o(3B)} \quad ,$$

$$C_{s32}^{3\omega} = I_o^2(B)\, I_1(B) - \frac{1}{6} - \left[I_o^3(B) - \frac{1}{2} \right] \frac{K_o(3A)\, I_1(3B) + I_o(3A)\, K_1(3B)}{K_o(3A)\, I_o(3B) - I_o(3A)\, K_o(3B)} \quad ,$$

$$C_{s31}^{\omega} = I_o^3(B) - 3I_1^2(B)\, I_o(B) + \frac{3I_1^3(B)}{B} - 3I_o^2(B)\, I_1(B) + 3I_1^3(B)$$

$$- \frac{15I_o(B)\, I_1^2(B)}{B} + \frac{12I_1^3(B)}{B^2} \; \frac{K_o(A)\, I_1(B) + I_o(A)\, K_1(B)}{K_o(A)\, I_o(B) - I_o(A)\, K_o(B)} \quad ,$$

$$C_{s32}^{\omega} = 3I_o^2(B)\, I_1(B) - I_c^3(B) \, \frac{K_o(A)\, I_1(B) + I_o(A)\, K_1(B)}{K_o(A)\, I_o(B) - I_o(A)\, K_o(B)} \quad .$$

The next step is to determine the polarization distances. From Equation (22) we have

$$v_{r3} = \frac{dP_{r3}}{dt} + \frac{\partial P_{r2}}{\partial r}\frac{dP_{r1}}{dt} + \frac{\partial P_{r2}}{\partial z}\frac{dP_{z1}}{dt} + \frac{\partial P_{r1}}{\partial r}\frac{dP_{r2}}{dt} + \frac{\partial P_{r1}}{\partial z}\frac{dP_{z2}}{dt}$$

$$+ \frac{dP_{r1}}{dt}\frac{\partial P_{r1}}{\partial z}\frac{\partial P_{z1}}{\partial r} + \frac{dP_{r1}}{dt}\left(\frac{\partial P_{r1}}{\partial r}\right)^2 + \frac{dP_{z1}}{dt}\frac{\partial P_{r1}}{\partial z}\frac{\partial P_{r1}}{\partial r} + \frac{dP_{z1}}{dt}\frac{\partial P_{z1}}{\partial z}\frac{\partial P_{r1}}{\partial z} \quad ,$$

and

$$v_{z3} = \frac{dP_{z3}}{dt} + \frac{\partial P_{z1}}{\partial r}\frac{dP_{r2}}{dt} + \frac{\partial P_{z1}}{\partial z}\frac{dP_{z2}}{dt} + \frac{\partial P_{z2}}{\partial r}\frac{dP_{r1}}{dt} + \frac{\partial P_{z2}}{\partial z}\frac{dP_{z1}}{dt}$$

$$+ \frac{\partial P_{z1}}{\partial r}\frac{\partial P_{r1}}{\partial z}\frac{dP_{z1}}{dt} + \frac{dP_{z1}}{dt}\left(\frac{\partial P_{z1}}{\partial z}\right)^2 + \frac{\partial P_{z1}}{\partial z}\frac{\partial P_{z1}}{\partial r}\frac{dP_{r1}}{dt} + \frac{dP_{r1}}{dt}\frac{\partial P_{r1}}{\partial r}\frac{\partial P_{z1}}{\partial r} \quad .$$

With these equations and Equations (51a-d), we may compute the third-order solutions for polarization distance. However, the expressions for the third-order polarization distance are rather complicated, and since they could not be correlated with experimental results, they are of little interest and will not be given. Graphic results for a typical case are presented in Figure 7.

As before, we find that the TE quantities are so small that they may be taken to be zero. Each of the third-order quantities has two parts, one with frequency ω and one with frequency 3ω. The terms with frequency ω add, of course, to the first-order solutions. We note that the results obtained thus far, when added, are series of the form,

$$M\left[f_1(r,z,t) + \sum_{n=2}^{3} \frac{n(M\beta_e)^n}{2^n} f_n(r,z,t)\right] \quad .$$

If we assume that the functions f_n all have some upper bound, the convergence of the solutions depend on the convergence of the series,

$$\sum_{n=2}^{\infty} \frac{n(M\beta_e)^n}{2^n} \quad .$$

This series converges for $M\beta_e < 2$, and we may conclude the $M\beta_e$ must be less than two for the results to be valid. However, the results should be used with caution for $M\beta_e > 1$, since there is no certainty that the series solution of the problem would have the form given above because the general term is not known. But we can say that the series will certainly be a power series of form,

$$f_1(r,z,t) + \sum_{n=2}^{3} C_n f_n(r,z,t)(M\beta_e)^n \quad ,$$

and should be uniformly convergent for $M\beta_e < 1$.

One further quantity is of interest. This is the total current density $\underline{J}$. We have

$$\underline{J}_n = 2\pi\left[\int_0^b \left(\sum_{k=0}^{n} \rho_{n-k}\, \underline{v}_k\right)\frac{dR}{\beta_e} + \underline{J}_{snb}\right] \quad , \qquad (52)$$

where

$$\underline{J}_{snb} = \sum_{k=1}^{n} \rho_{sk} \, \underline{v}_{n-k} \Bigg|_{R=B}$$

We now consider the results of the analysis.

CONCLUSIONS AND RECOMMENDATIONS

The results of the preceding sections are shown graphically in Figures 7 through 11. The beam geometry was assumed to be such that $B/A = .5$ and $B = .5$, and $M\beta_e$ was taken to be equal to one. This is a reasonable value of beam geometry, and $M\beta_e$ was chosen to be large enough to show the nonlinear effects well.

Figure 7 shows the longitudinal polarization distances. It may be observed that the third-order polarization distance is quite large. The amplitude of the fundamental polarization distance is strongly influenced by the correction term obtained from the third-order solutions. An interesting detail may be observed about the second-order polarization distance. The amplitude of this quantity never changes sign, so the polarization distance is always in the same direction. This effect may also be noted in Figure 5, where the edge radius of the beam is always decreased by the R-F disturbance. This effect is strongly related to the choice of modulation conditions made here and might not be observed if different modulation conditions were used.

Figures 8 and 9 show the velocity modulation effect. The longitudinal velocity is almost periodic in appearance and the fundamental component is not too strongly influenced by the higher-order solutions. The velocity is shown for the case of electrons at the center of the beam. For relatively thin beams there will not be much variation in the relative amplitude of the components as one moves from center to edge. The radial velocity at the edge of the beam is completely aperiodic and shows extreme fluctuations.

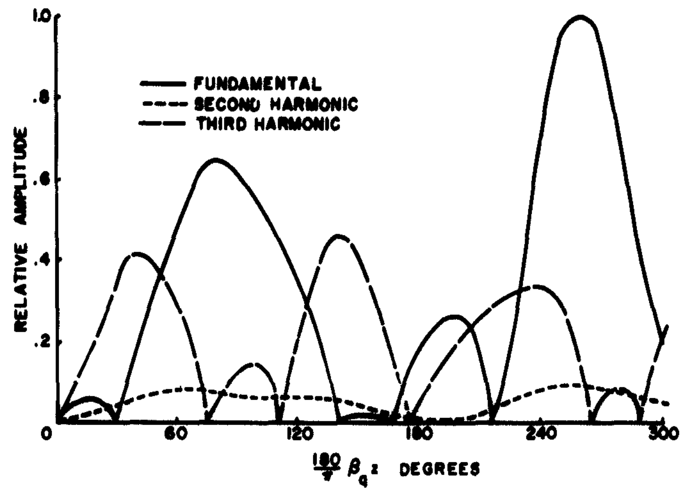

Figure 7. Amplitude of Longitudinal Polarization Distance P_z at Center of Beam.

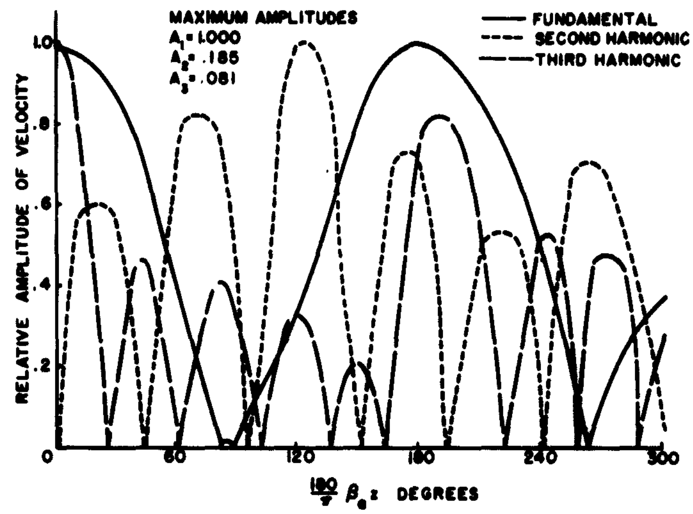

Figure 8. Longitudinal Component of Velocity at Center of Beam.

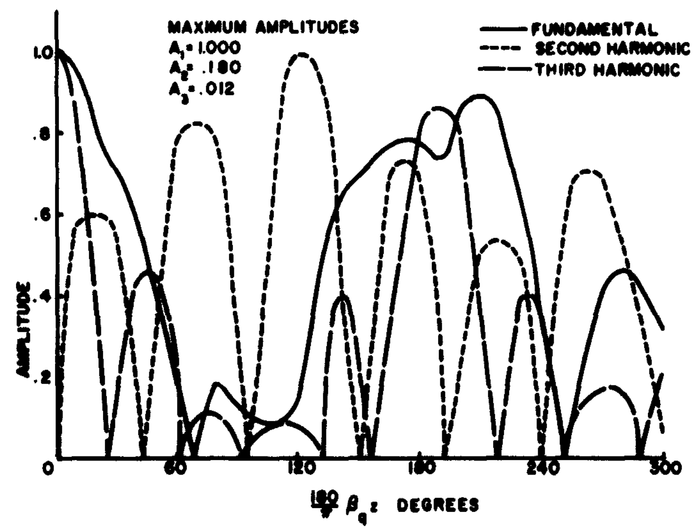

Figure 9. Amplitude of Radial Velocity at Edge of Beam.

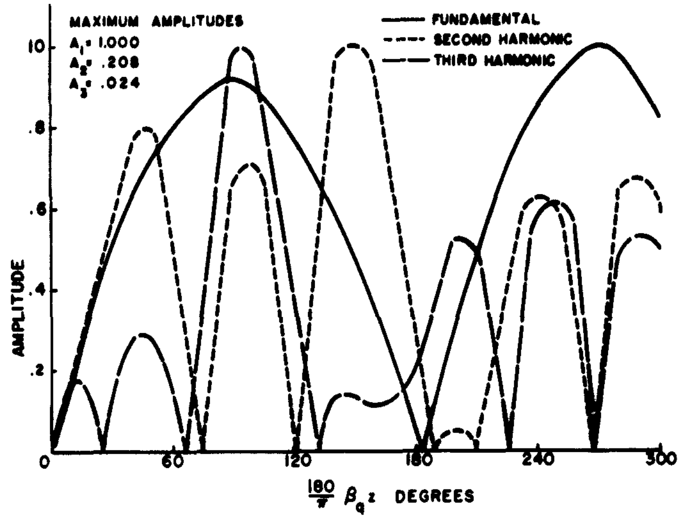

Figure 10. Total Current in Longitudinal Direction.

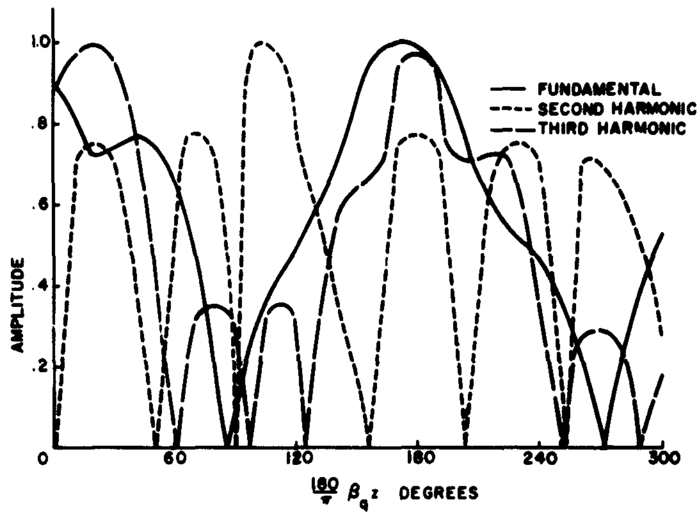

Figure 11. Amplitude of Total Radial Current.

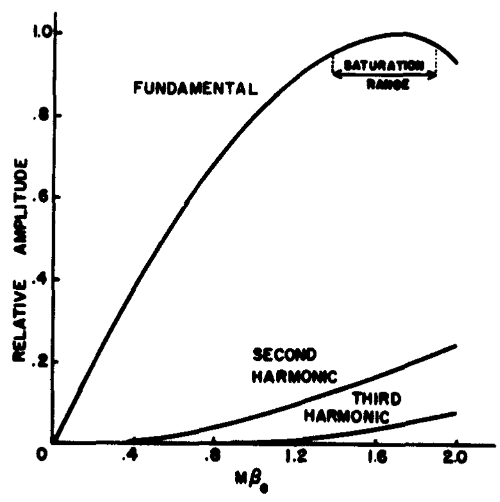

Figure 12. Amplitude of Axial RF Currents at $\frac{1}{4}$ Reduced Plasma Wavelength from Input (as a function of modulation parameter M, showing current saturation).

Of most interest is the current density, as this is strongly related to the output voltage to be expected if an output gap is placed at some point following the input gap. If we imagine that an electron moves through a gap in a drift tunnel, we see at once that it will induce a charge on the edges of the gap and therefore produce an electric field across the gap. Naturally, the strength of such a field would be proportional to the number of electrons producing the induced charge. This rough example gives an idea of the relation between the induced electric field and the current density in the gap. If mathematical expression of the relation is desired, it may be deduced from the fact that

$$\underline{J}_T = \underline{J}_c + \epsilon \frac{\partial \underline{\underline{E}}}{\partial t} \quad ,$$

where $\underline{J}_T$ is the total current and $\underline{J}_c$ the conduction current.

Figures 10 and 11 show the amplitude of the components of current density at various distances from the input gap. It is found that the radial current density is of the order β_q/β_e less than the axial current density and hence constitutes a very small portion of the total current.

Most of the fundamental current density is produced by the surface current. In the higher-order solutions, the current is produced by both conduction and surface effects. The higher-order conduction currents arise from the density modulation of the beam rather than from the velocity modulation.

The contribution of the third-order correction term to the fundamental is most significant. It is found that the voltage across a hypothetical output gap placed one-quarter-plasma wavelength from the input will increase with increasing input voltage, but if the input voltage is made large enough no further increase will be obtained. This effect is shown by Figure 12 and indicates a method of applying the results to the problem of current saturation in klystrons. The amplitudes of the second and third harmonics are shown for comparison.

It will be noted that the higher-order solutions apparently tend to become infinite if $\beta_p = 2\beta_q$ or $\beta_p = 3\beta_q$. Actually this is not the case. If one of the these conditions is true, a driving frequency will be equal to a natural resonance of the system and growth of the waves result. This is a sort of parametric amplification and has been observed in actual beams by Mihran.[24] For the case $\beta_p = 2\beta_q$ we would have, for example,

$$E_{z2} = \frac{M^2 \beta_e}{16\eta} \left\{ \left[I_o^2(R) - I_1^2(R) \right] \left[1 - \beta_p z \sin\beta_p z - \cos\beta_p z \right] \sin 2\phi \right\} \quad .$$

A plot of the axial current density is given for a case in which $\beta_p = 2\beta_q$ in Figure 13.

This effect of harmonic current growth under appropriate conditions obviously would have important implications for the design of frequency multiplier klystrons. The theory of the confined beam developed by Paschke[14, 15, 16] predicts such behavior only for very thin beams, but since the Brillouin beam has a property of possessing a natural resonance at the plasma frequency, it should be possible to obtain growth in the amplitude of the harmonic with any beam geometry.

The present work also can be applied to the suppression of outputs of unwanted higher harmonics in klystrons. It will be noted from Figure 10 that both second- and third-harmonic current amplitudes are small at

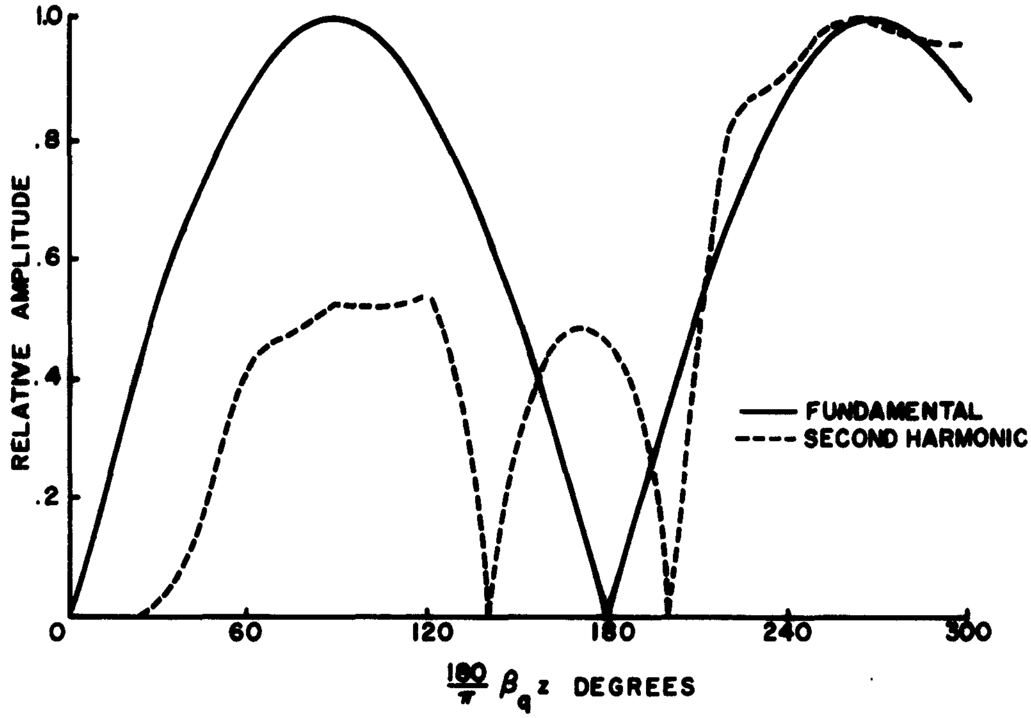

Figure 13. Axial Current Density if $2\beta_q = \beta_p$ Showing Growth of Second Harmonic.

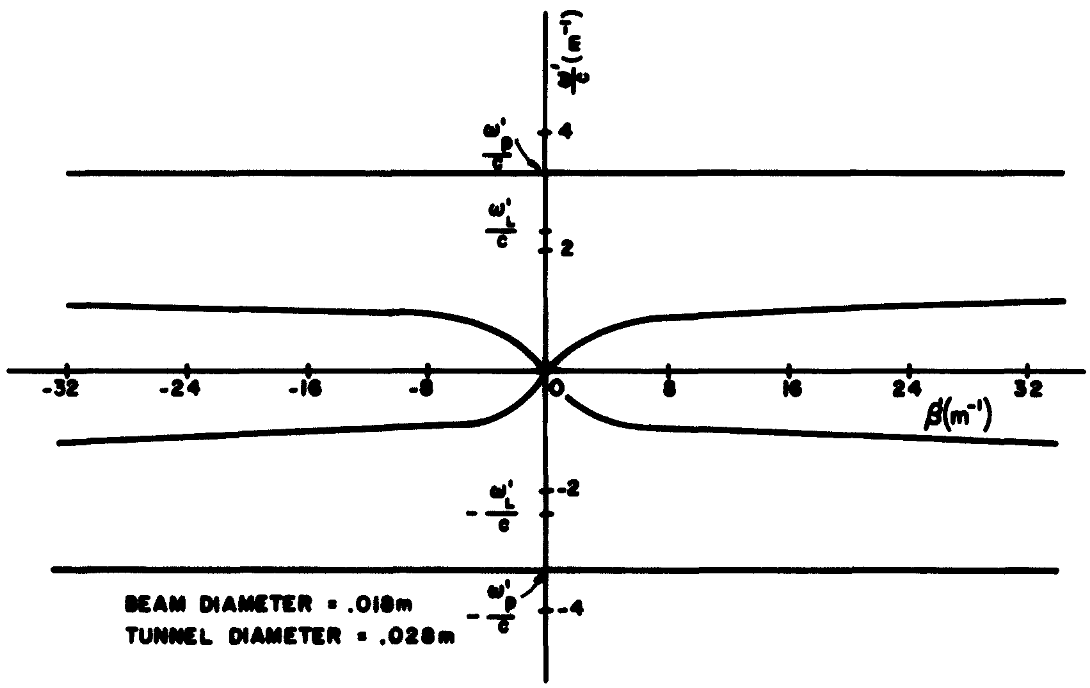

Figure 14. Solutions for ω' and β' for a Typical Case.

$180/\pi\,\beta_q z = 70^{\circ}$, while the fundamental component is quite large. This would be an ideal place to put an output gap, if it were important to suppress the output of second and third harmonics.

This analysis leaves several problems unsolved. The most important is the problem of satisfying the boundary conditions at the edge of the beam. The method employed here forces the solutions to satisfy the boundary conditions by making an appropriate choice for the surface charge density. However, there is some doubt as to the validity of this step. Perhaps further research will shed some light on the solution of this problem.

A nonlinear analysis based on the small-signal analysis of Paschke[13] failed to give a convergent series of functions as a solution. The basic equations used in this analysis are believed to be correct therefore, and the approach used here should be a good method for further work since it is relatively straightforward. The problem of the extension of this analysis to a beam with more general focusing system, however, would be a most difficult one. Studies of electron motion in gaps will also be useful to give a better description of the initial conditions.

This study is believed to be the closest approach yet made to the description of the behavior of an actual device, since the effects of a finite magnetic focusing field are taken into account, and it is hoped that the results of this study will be useful in predicting the behavior of actual devices. There is still much work to be done, however, before we can obtain an accurate picture of the large-signal behavior of real devices.

APPENDIX A: GLOSSARY OF SYMBOLS

Roman Letter Symbols

a	=	radius of drift tunnel
A	=	$\beta_e a$
A_1, A_2	=	constants
b	=	radius of unperturbed beam
B	=	$\beta_e b$
B_r	=	radial magnetic-field strength
B_z	=	longitudinal magnetic-field strength
B_θ	=	azimuthal magnetic-field strength
B_o	=	focusing magnetic-field strength
c	=	velocity of light
C	=	constant
d	=	width of modulating gap
D	=	$\beta_e D$
e	=	electronic charge
E	=	electric field
F, f	=	notation for functions
G, f	=	notation for functions
H	=	normalized magnetic-field strength, $\frac{4\omega_L}{\omega}$
I_o, I_1	=	modified Bessel function of first kind
J	=	current density
K_o, K_1	=	modified Bessel function of second kind
k_m	=	magnetic constant
M	=	gap-coupling coefficient

m	=	electron mass
P	=	polarization distances
Q	=	drive terms
r	=	radial co-ordinate
R	=	$\beta_e r$
t	=	time
T	=	ωt
v	=	velocity
z	=	longitudinal co-ordinate
Z	=	$\beta_e z$

<u>Greek Letter Symbols</u>

α	=	modulation coefficient
β	=	phase constant
β_e	=	ω/v_{zo}
β_p	=	ω_p/v_{zo}
β_q	=	ω_q/v_{zo}
γ, Γ	=	radial propagation constants
ϵ_o	=	permittivity of free space
η	=	e/m
θ	=	azimuthal co-ordinate
μ_o	=	permeability of free space
ξ	=	normalized velocity
ρ	=	charge density
ρ_s	=	surface charge density

ϕ	=	T - Z
ω	=	fundamental frequency
ω_L	=	Larmor frequency, $\frac{\eta B_o}{2}$
ω_p	=	plasma frequency
ω_q	=	reduced plasma frequency

APPENDIX B: THEORY OF THE RELATIVISTIC BRILLOUIN BEAM

We now seek to extend our analysis to include beams in which the d-c velocity v_{zo} does not satisfy the condition $v_{zo} \ll c$.

To facilitate the analysis we choose a co-ordinate system moving with respect to the structures surrounding the beam with the velocity v_{zo} and in the same direction as the electron stream. As before we refer to the co-ordinates described by Figure 3 of the text as r, θ, z, t, the rest system, and denote the co-ordinates of this system as r', θ', z', t', the beam system. We restrict our attention to the Brillouin beam.

The equations of motion for small-signal solution in the beam system may be written as

$$\frac{\partial^2 P'_{r1}}{\partial t'^2} = \eta E'_{r1} - \eta r \omega_L B'_{z1} \quad ,$$

$$\frac{\partial^2 P'_{\theta 1}}{\partial t'^2} = \eta E'_{\theta 1} \quad ,$$

$$\frac{\partial^2 P'_{z1}}{\partial t'^2} = \eta E'_{z1} + \eta r \omega_L B'_{r1} \quad ,$$

We proceed to derive a wave equation exactly as in the first-order solution in Chapter II. We neglect TE quantities, combine with the Maxwell equations, and arrive at the following result:

$$\left(\frac{\partial^2}{\partial t'^2} + \omega_p^2\right)\left[\frac{1}{r'}\frac{\partial}{\partial r'} r' \frac{\partial}{\partial r'} + \frac{\partial^2}{\partial^2 z'} - \frac{1}{c^2}\frac{\partial^2}{\partial^2 t} + \frac{\omega_p^2}{c^2}\right] P'_z = 0 \qquad \text{(B.1)}$$

It will be noted that these equations are the same as those of the main text except for the replacement of the d/dt of the text by $\partial/\partial t'$. The reason for this is, of course, that $\partial/\partial t + v_{zo}\ \partial/\partial z = d/dt$, and v_{zo} appears to be zero in the moving co-ordinate system.

The boundary conditions are matched in the same manner as in the main text and we find two possible sets of solutions to Equation (B.1) for space-charge waves after conditions for propagation are considered:

$$P'^{(1)}_{z'} = I_o(\gamma' r') \exp\left[j(\omega_p t' - \beta'^{(1)} z')\right] \tag{B.2}$$

and

$$P'^{(2)}_{z'} = \exp\left[j(\omega_p t' - \beta'^{(2)} z')\right] \quad , \tag{B.3}$$

where

$$\frac{1}{\gamma' b'}\left(1 - \frac{\omega_p^2}{\omega'^2}\right)\frac{I_1(\gamma' b')}{I_o(\gamma' b')} = \frac{1}{\gamma' b'}\left\{\frac{K_o(\Gamma' a')\, I_1(\Gamma' b') + I_o(\Gamma' a')\, K_1(\Gamma' b')}{K_o(\Gamma' a')\, I_o(\Gamma' b') - I_o(\Gamma' a')\, K_o(\Gamma' b')}\right\} , \tag{B.4}$$

and

$$\Gamma'^2 = \left[\beta'^{(1)}\right]^2 - \frac{\omega'^2}{c^2} \quad ,$$

$$\gamma'^2 = \Gamma'^2 + \frac{\omega_p^2}{c^2} \quad .$$

The values of $\beta'^{(1)}$ and ω' are fixed by Equation (B.4); $\beta'^{(2)}$ may assume any value. Figure 14 shows the appearance of a typical set of solutions for ω' and β'.

The next step is to transform the solutions (B.2) and (B.3) into solutions useful in the rest system. The equations of transformation are:

$$z = \frac{1}{\sqrt{1 - \frac{v_{zo}^2}{c^2}}} \quad (z' - v_{zo} t') \tag{B.5a}$$

$$r = r' \tag{B.5b}$$

$$\theta = \theta' \tag{B.5c}$$

$$t = \frac{1}{\sqrt{1 - \frac{v_{zo}^2}{c^2}}} \left(t' - \frac{v_{zo} z'}{c^2} \right) \quad . \tag{B.5d}$$

Further, if we have a wave described by

$$e^{j(\omega' t' - \beta' z')}$$

in the beam system, we find that the same wave is described by

$$e^{j\left(\frac{\omega' + v_{zo}\beta'}{\sqrt{1 - \frac{v_{zo}^2}{c^2}}} t + \frac{\beta' + \frac{v_{zo}\omega'}{c^2}}{\sqrt{1 - \frac{v_{zo}^2}{c^2}}} z \right)}$$

in the rest system, by straightforward application of Equations (B.5).

Thus, we conclude:

$$\beta = \frac{\beta' + \frac{v_{zo}\omega'}{c^2}}{\sqrt{1 - \frac{v_{zo}^2}{c^2}}} \quad , \tag{B.6a}$$

$$\omega = \frac{\omega' + v_{zo}\beta'}{\sqrt{1 - \frac{v_{zo}^2}{c^2}}} \quad . \tag{B.6b}$$

The equations of transformation for the electromagnetic fields are

$$E_z = E_z' \quad , \tag{B.7a}$$

$$E_r = \frac{1}{\sqrt{1 - \frac{v_{zo}^2}{c^2}}} \left(E_r' + v_{zo} B_\theta'\right) \quad , \tag{B.7b}$$

$$E_\theta = \frac{1}{\sqrt{1 - \frac{v_{zo}^2}{c^2}}} \left(E_\theta' - v_{zo} B_r'\right) \quad , \tag{B.7c}$$

$$B_z = B_z' \tag{B.7d}$$

$$B_r = \frac{1}{\sqrt{1 - \frac{v_{zo}^2}{c^2}}} \left(B_r' - \frac{v_{zo}}{c^2} E_\theta'\right) \tag{B.7e}$$

$$B_\theta = \frac{1}{\sqrt{1 - \frac{v_{zo}^2}{c^2}}} \left(B_\theta' + \frac{v_{zo}}{c^2} E_r'\right) \tag{B.7f}$$

$$\rho = \frac{\rho'}{\sqrt{1 - \frac{v_{zo}^2}{c^2}}} \left(1 - \frac{\tilde{v}_z' v_{zo}}{c^2}\right) \quad . \tag{B.7g}$$

The velocities transform as follows:

$$\tilde{v}_z = \frac{\tilde{v}'_z}{1 - \frac{\tilde{v}'_z v_{zo}}{c^2}} \doteq \tilde{v}'_z \left(1 - \frac{v_{zo}^2}{c^2}\right) \tag{B.8a}$$

$$\tilde{v}_{r,\theta} = \frac{\tilde{v}'_{r,\theta}}{1 - \frac{\tilde{v}_z v_{zo}}{c^2}} \sqrt{1 - \frac{v_{zo}^2}{c^2}} \doteq \tilde{v}'_{r,\theta} \sqrt{1 - \frac{v_{zo}^2}{c^2}} \quad . \tag{B.8b}$$

With the equations of transformation (B.5), (B.6), (B.7), and (B.8), we find the following solutions valid in the rest system:

$$P_{z1} = I_o(\gamma r)\; e^{j(\omega t - \beta z)} \tag{B.9a}$$

$$P_{r1} = \frac{j\left(\beta - \frac{\omega v_{zo}}{c^2}\right)}{\gamma\left(1 - \frac{v_{zo}^2}{c^2}\right)}\; I_1(\gamma r)\; e^{j(\omega t - \beta z)} \tag{B.10a}$$

$$v_{z1} = j(\omega\beta v_{zo})\; I_o(\gamma r)\; e^{j(\omega t - \beta z)} \tag{B.9c}$$

$$v_{r1} = -\frac{\left(\omega - \beta v_{zo}\right)\left(\beta \frac{\omega v_{zo}}{c^2}\right)}{\gamma\left(1 - \frac{v_{zo}^2}{c^2}\right)}\; I_1(\gamma r)\; e^{j(\omega t - \beta z)} \tag{B.9d}$$

$$\mathcal{P}_1 = \frac{j v_{zo}\, \omega_p^2 (\omega - \beta v_{zo})}{\eta c^2 \left(1 - \frac{v_{zo}^2}{c^2}\right)} \, I_o(\gamma r)\, e^{j(\omega t - \beta z)} \tag{B. 9e}$$

$$P_\theta = v_\theta = B_r = B_z = E_\theta = 0 \quad . \tag{B. 9f}$$

In Equations (B. 9)

$$\gamma = \sqrt{\beta^2 - \frac{\omega^2}{c^2} + \frac{\omega_p^2}{c^2}} \tag{B. 10a}$$

$$\omega = \frac{\omega' - \beta' v_{zo}}{\sqrt{1 - \frac{v_{zo}^2}{c^2}}} \tag{B. 10b}$$

$$\beta = \frac{\beta' - \frac{v_{zo}\,\omega'}{c^2}}{\sqrt{1 - \frac{v_{zo}^2}{c^2}}} \quad . \tag{B. 10c}$$

Equations (B. 9) give the solutions in the rest system corresponding to Equation (B. 2). Waves in the rest system corresponding to Equation (B. 3) have the form $e^{j(\omega t - \beta z)}$, where

$$\beta = \frac{\beta' - \frac{v_{zo}\,\omega_p}{c^2}}{\sqrt{1 - \frac{v_{zo}^2}{c^2}}} \tag{B. 11a}$$

and

$$\omega = \frac{\omega_p - \beta' v_{zo}}{\sqrt{1 - \frac{v_{zo}^2}{c^2}}} \quad . \tag{B.11b}$$

It should be noted that ω_p is the same in both co-ordinate systems.

To determine the phase constants for the waves, we use the following procedure: A curve of $\omega' - \beta'$ similar to Figure 14 is calculated, and the value of ω in the rest system is specified. Then the values of ω' and β' are determined from the diagram so that Equations (B. 10b) and (B. 11b) give the correct value of ω in the rest system. The values of β in the rest system may then be found from Equations (B. 10c) and (B. 11a). There will be two waves of each type for which propagation is possible.

In general, each of the two waves in the beam system corresponding to Equation (B. 2) will have different frequencies as well as different phase constants. This fact would make it very difficult to apply the method described in Chapter IV to get a large-signal solution. Probably the best method of approach to the problems of large-signal solution and initial conditions would be through use of a computer.

APPENDIX C: NONLINEAR EQUATIONS OF MOTION IN TERMS OF THE POLARIZATION DISTANCES

It may be useful for some purposes to have the nonlinear equations of motion stated in terms of the polarization distances rather than the velocities. These are given as follows:

<u>First-Order Equations</u>:

$$\frac{d^2}{dt^2} P_{r1} + 2k_m \omega_L \frac{dP_{\theta 1}}{dt} = \eta E_{r1} + \eta r \omega_o B_{z1} - \eta v_{zo} B_{\theta 1} \qquad \text{(C. 1)}$$

and

$$\frac{d^2}{dt^2} P_{\theta 1} - 2k_m \omega_L \frac{dP_{r1}}{dt} = \eta E_{\theta 1} + \eta v_{zo} B_{r1} \qquad \text{(C. 2)}$$

and

$$\frac{d^2}{dt^2} P_{z1} = \eta E_{z1} - \eta r \omega_o B_{r1} \qquad \text{(C. 3)}$$

<u>Second-Order Equations</u>:

$$\frac{d^2 P_{r2}}{dt^2} + 2k_m \omega_L \frac{dP_{\theta 2}}{dt} + \frac{\partial P_{z1}}{\partial z} \frac{d^2 P_{z1}}{dt^2} + \frac{\partial P_{r1}}{\partial r} \frac{d^2 P_{r1}}{dt^2} + \frac{dP_{r1}}{dt} \frac{\partial}{\partial r} \frac{dP_{r1}}{dt}$$

$$+ 2 \frac{dP_{z1}}{dt} \frac{\partial}{\partial z} \frac{dP_{r1}}{dt} - \frac{1}{r} \left(\frac{dP_{\theta 1}}{dt}\right)^2 + 2k_m \omega_L \frac{\partial P_{\theta 1}}{\partial z} \frac{dP_{z1}}{dt}$$

$$+ 2k_m \omega_L \left(\frac{\partial P_{\theta 1}}{\partial r} - \frac{P_{\theta 1}}{r}\right)\left(\frac{dP_{r1}}{dt}\right) = \eta E_{r2} + \eta r \omega_o B_{z2} + \eta \frac{dP_{\theta 1}}{dt} B_{z1}$$

$$- \eta \frac{dP_{z1}}{dt} B_{\theta 1} - \eta v_{zo} B_{\theta 2} \qquad \text{(C. 4)}$$

and

$$\frac{d^2P_{\theta 2}}{dt^2} - 2k_m\omega_L \frac{dP_{r2}}{dt} + 2\frac{dP_{z1}}{dt}\frac{\partial}{\partial z}\frac{dP_{\theta 1}}{dt} + 2\frac{dP_{r1}}{dt}\frac{\partial}{\partial r}\frac{dP_{\theta 1}}{dt} + \frac{\partial P_{\theta 1}}{\partial z}\frac{d^2P_{z1}}{dt^2}$$

$$+ \left(\frac{d^2P_{r1}}{dt^2}\frac{\partial P_{\theta 1}}{\partial r}\right) - \frac{P_{\theta 1}}{r} - 2k_m\omega_L \frac{\partial P_{r1}}{\partial z}\frac{\partial P_{z1}}{dt} - 2k_m\omega_L \frac{\partial P_{r1}}{\partial r}\frac{\partial P_{r1}}{dt}$$

$$= \eta E_{\theta 2} + \eta \frac{dP_{z1}}{dt} B_{r1} - \eta \frac{dP_{r1}}{dt} B_{z1} + \eta v_{zo} B_{r1} \qquad \text{(C. 5)}$$

and

$$\frac{d^2P_{z2}}{dt^2} + \frac{\partial P_{z1}}{\partial z}\frac{d^2P_{z1}}{dt^2} + \frac{\partial P_{z1}}{\partial r}\frac{d^2P_{r1}}{dt^2} + 2\frac{dP_{z1}}{dt}\frac{\partial}{\partial z}\frac{dP_{z1}}{dt} + 2\frac{dP_{r1}}{dt}\frac{\partial}{\partial r}\frac{dP_{z1}}{dt}$$

$$= \eta E_{z2} - \eta r\omega_o B_{r2} + \eta \frac{dP_{r1}}{dt} B_{\theta 1} - \eta \frac{dP_{\theta 1}}{dt} B_{r1} \quad . \qquad \text{(C.6)}$$

The third-order equations are very involved, and probably too difficult to use in practical calculations, so we do not include them. It may be seen that application of Equations (C. 4), (C. 5) and (C. 6) to the problem of finding a second-order solution would be more difficult than the method of Chapter IV.

APPENDIX D: SMALL SIGNAL SOLUTIONS FOR ARBITRARY VALUES OF MAGNETIC FOCUSING FIELD

If we assume that magnetic forces on the electrons are small compared to the electric forces (and this will be the case for nonrelativistic beams), we may write the equations of motion as follows:

$$\frac{dv_{r1}}{dt} + 2k_m \omega_L \dot{v}_{\theta 1} = \eta E_{r1} \quad , \tag{D. 1}$$

$$\frac{dv_{\theta 1}}{dt} - 2k_m \omega_L v_{r1} = \eta E_{\theta 1} \quad , \tag{D. 2}$$

$$\frac{dv_{z1}}{dt} = \eta E_{z1} \quad . \tag{D. 3}$$

We now combine Equations (D. 1), (D. 2) and (D. 3) with the Maxwell equations, assuming that beam scalloping is small so that $v_{ro} = 0$. The assumption is made that $E_{\theta 1}$ is small so that Equations (D. 1) and (D. 2) may be combined in the form,

$$\frac{d^2 v_{r1}}{dt^2} + 4k_m^2 \, \omega_L^2 \, v_{r1} = \eta \, \frac{dE_{r1}}{dt} \quad . \tag{D. 4}$$

Then Equations (D. 3) and (D. 4) are combined with Equations (8c), (9a), and (9b) to give the following results:

$$v_{z1} = I_o(\gamma_b r) \, e^{j(\omega t - \beta z)} \quad , \tag{D. 5}$$

$$v_{r1} \doteq \frac{j\gamma_b}{\beta_e}\left(\frac{1}{1 - \dfrac{4k_m^2\,\omega_L^2}{\omega_q^2}}\right) I_1(\gamma_b r)\, e^{j(\omega t - \beta z)} , \qquad (D.6)$$

where

$$\gamma_b^2 \doteq \frac{\beta_e^2 \left(1 - \dfrac{\omega_p^2}{\omega_q^2}\right)\left[4k_m^2\,\omega_L^2 - \omega_q^2\right]}{\left[4k_m^2\,\omega_L^2 + \omega_p^2 - \omega_q^2\right]} , \qquad (D.7)$$

and $\beta = \beta_e \pm \beta_q$ as before. For purposes of matching at the beam boundary we have

$$\frac{1}{\gamma_b b}\left(1 - \frac{\omega_p^2}{\omega_q^2 - 4k_m^2\,\omega_L^2}\right)\frac{I_1(\gamma_b b)}{I_o(\gamma_b b)} = \frac{1}{\Gamma b}\left[\frac{K_o(\Gamma a)\, I_1(\Gamma b) + I_o(\Gamma a)\, K_1(\Gamma b)}{K_o(\Gamma a)\, I_o(\Gamma b) - I_o(\Gamma a)\, K_o(\Gamma b)}\right] ,$$

(D.8)

where $\Gamma^2 \doteq \beta_e^2$.

So far the solution is comparatively uncomplicated. However, we also have

$$v_\theta = \pm \frac{2k_m\omega_L\gamma_b}{\omega_q\beta_e}\left(\frac{1}{1 - \dfrac{4k_m^2\,\omega_L^2}{\omega_q^2}}\right) I_1(\gamma_b r)\, e^{j(\omega t - \beta z)} . \qquad (D.9)$$

It is the presence of the velocity in the θ-direction which makes the solution difficult. As may be seen from Equations (D.7) and (D.9), if $2k_m\omega_L \ll \omega_q$, our results for the Brillouin beam are still very good approximations. However, if this is not the case, then the existence of a velocity in the θ direction makes a nonlinear solution much more difficult to obtain.

The quantity $E_{\theta 1}$ is also of interest in this analysis. We apply the previous results and Equation (9b) to find:

$$\eta E_{\theta 1} = -\frac{j\omega_q v_o}{\gamma_b c^2}\left\{\omega_o \gamma_b r\, I_o(\gamma_b r) + \left[\frac{2k_m \omega_L \omega_p^2}{\left(4k_m^2 \omega_L^2 - \omega_q^2\right)\left(1 - \frac{\beta_e^2}{\gamma_b^2}\right)} - \frac{2\omega_o}{\left(1 - \frac{\beta_e^2}{\gamma_b^2}\right)}\right] I_1(\gamma_b r)\right\} e^{j(\omega t - \beta z)} \quad . \tag{D. 10}$$

Because of the factor $1/c^2$ in the first coefficent, the value of Equation (D. 10) will always be small, as assumed above.

To sum up, we see that the analysis of this section shows that the analysis of the Brillouin beam may be applied when k_m is small. If k_m is not small a nonlinear solution will be most difficult to obtain, and the analyses made of the confined beam may be of considerable use.

APPENDIX E: VALUES OF CONSTANTS USED IN THE THIRD-ORDER SOLUTION

$$C_1 = \frac{3R_1^4}{(1-4R_1^2)(1-9R_1^2)}$$

$$C_2 = \frac{R_1^4}{(1-4R_1^2)(1-R_1^2)}$$

$$C_3 = \frac{1-2R_1^2}{1-4R_1^2} \; \frac{1-2R_1+R_1^2}{(2-R_1)\;1-\frac{1}{R_1}}$$

$$C_4 = \frac{1-2R_1^2}{1-4R_1^2} \; \frac{1-2R_1-R_1^2}{(2+R_1)\;1+\frac{1}{R_1}}$$

$$C_5 = \frac{3R_1^2-8R_1^4}{(1-R_1^2)(1-4R_1^2)}$$

$$C_6 = \frac{R_1^2}{(1-4R_1^2)(1-9R_1^2)}$$

$$C_7 = -\frac{1-2R_1^2}{(1-4R_1^2)(2-R_1)}$$

$$C_8 = \frac{1-2R_1^2}{(1-4R_1^2)(2+R_1)}$$

$$C_9 = \frac{R_1^2}{(1-4R_1^2)(1-9R_1^2)}$$

$$C_{10} = \frac{R_1^2}{(1-4R_1^2)(1-R_1^2)}$$

$$C_{11} = -\frac{1-2R_1^2}{(1-4R_1^2)(2-R_1)}$$

$$C_{12} = \frac{1-2R_1^2}{(1-4R_1^2)(2+R_1)}$$

$$C_{13} = \frac{3R_1^2-8R_1^4}{(1-R_1^2)(1-4R_1^2)}$$

$$C_{14} = \frac{3R_1^2}{(1-4R_1^2)(1-9R_1^2)}$$

$$C_{15} = -\frac{(1+R_1)(1-2R_1^2)}{(1-4R_1^2)(2-R_1)}$$

$$C_{16} = \frac{(1-R_1)(1-2R_1^2)}{(1-4R_1^2)(2+R_1)}$$

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Sven Olving

School of Electrical Engineering
CORNELL UNIVERSITY
Ithaca, New York

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THE KINETIC A-C POWER FLOW IN NONHOMOGENEOUS RELATIVISTIC ELECTRON BEAMS

S. Olving

LINEAR BEAM MICROWAVE TUBES

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ABSTRACT

This note discusses kinetic power flow and kinetic energy in relativistic space-charge-wave devices. The electrons are constrained to move in the z-direction only. The average electron velocity is assumed to be an arbitrary function of z and the transverse co-ordinates.

I. INTRODUCTION

Practically all low-level longitudinal-oscillation electron-beam devices, such as h-f diodes, klystrons, traveling-wave tubes, etc., are best described and understood in terms of linearized space-charge waves.[1] In addition to the electromagnetic power flow, given by Poynting's vector, these waves are associated with a kinetic power flow. To take advantage of the power-conservation principle, one has to take the kinetic power flow into account. In doing so, however, one finds that cross products between the wave quantities have to be included in order to get a finite time-average a-c power flow. Such cross products are ignored in the linearized space-charge-wave theory, and one is therefore inclined to believe that a first-order nonlinear space-charge-wave theory is necessary in order to formulate expressions for the kinetic power flow and the kinetic energy.

It was first realized by Chu[2] that the kinetic space-charge-wave power flow, can, in fact, be expressed in terms of the linearized space-charge-wave quantities in spite of the fact that the expression will contain cross products that are of the same order as the first-order nonlinear quantities that are ignored. The expressions and concepts developed by Chu have led to many important discoveries concerning the flow of signals and noise in electron beams.

An elegant treatment of the kinetic power flow has been given by Louisell and Pierce.[3] These authors deal with the case of a nonrelativistic electron beam with constant average drift velocity. The purpose of the present note is to construct expressions for the kinetic power flow and the kinetic energy in a relativistic beam with arbitrarily varying drift velocity.

II. BASIC RELATIONS

To begin with let us deduce a general relation which can be obtained from Maxwell's equations:

$$\text{curl } \bar{H} = \bar{i} + \epsilon \frac{\partial \bar{E}}{\partial t} \quad , \tag{1}$$

$$\text{curl } \bar{E} = -\mu \frac{\partial \bar{H}}{\partial t} \quad . \tag{2}$$

If one multiplies Equation (1) by $\bar{E}$, Equation (2) by $\bar{H}$, and subtracts the two equations, one obtains

$$\boxed{\text{div } \bar{P}_{em} + \frac{\partial}{\partial t} W_{em} + \bar{i} \cdot \bar{E} = 0} \quad , \tag{3}$$

where $\bar{P}_{em} = \bar{E} \times \bar{H}$ is the Poynting vector while $W_{em} = \frac{1}{2} \epsilon \bar{E}^2 + \frac{1}{2} \mu \bar{H}^2$ is the electromagnetic energy density. It should be pointed out that we have made use of the vector formula,

$$\bar{a} \cdot \text{curl } \bar{b} - \bar{b} \cdot \text{curl } \bar{a} + \text{div } (\bar{a} \times \bar{b}) = 0$$

in deducing Equation (3).

In addition to the electromagnetic energy and power, we have a kinetic energy density W_k and a kinetic power-flow density $\bar{P}_k$ associated with the electron beam. Since the total energy is conservative, we have the continuity equation,

$$\text{div } (\bar{P}_{em} + \bar{P}_k) + \frac{\partial}{\partial t} (W_{em} + W_k) = 0 \quad . \tag{4}$$

From Equations (3) and (4), one expresses

$$\boxed{\bar{i} \cdot \bar{E} = \operatorname{div} \bar{P}_k + \frac{\partial W_k}{\partial t}} \quad . \tag{5}$$

We are assuming that the electron motion is constrained to the z-direction only. Thus i and P_k have no components in the transverse direction and Equation (5), which like relation (4) describes energy conservation, can be written in the form,

$$i_z E_z = \frac{\partial P_{kz}}{\partial t} + \frac{\partial W_k}{\partial t} \quad , \tag{6}$$

where i_z is the total (dc + ac) electron current density of the beam.

If the total kinetic energy of the electron is w, one has

$$dw = - eE_z dz \quad \text{or} \quad \frac{dw}{dz} = - eE_z \quad . \tag{7}$$

Furthermore, one has the relations (ρ = electronic charge density),

$$W_k = \frac{\rho}{-e} w \tag{8}$$

and

$$P_{kz} = \frac{i_z}{-e} w \quad . \tag{9}$$

By the use of Equations (7), (8), and (9), the continuity equation $(\partial i_z/\partial z) + (\partial \rho/\partial t) = 0$ and the relation $i_z = v_z \rho$, one obtains

$$\frac{\partial P_{kz}}{\partial z} + \frac{\partial W_k}{\partial t} = -\frac{1}{e}\left[\frac{\partial i_z}{\partial z} w + i_z \frac{\partial w}{\partial z} + \frac{\partial \rho}{\partial t} w + \rho \frac{\partial w}{\partial t}\right]$$

$$= -\frac{1}{e} i_z \left[\frac{\partial w}{\partial z} + \frac{1}{v_z} \frac{\partial w}{\partial t}\right] = -\frac{i_z}{e} \frac{dw}{dz} = i_z E_z \quad , \tag{10}$$

which proves Equation (6).

III. KINETIC POWER AND ENERGY ASSOCIATED WITH THE LINEAR SPACE-CHARGE WAVES

We now write

$$i_z = i_o + i_1 + i_2 \qquad \rho = \rho_o + \rho_1 + \rho_2 \qquad ,$$

$$v_z = v_o + v_1 + v_2 \qquad E_z = E_o + E_1 + E_2 \qquad ,$$

where the subscripts o, 1, 2 denote terms of the zeroth (i. e., undisturbed d-c terms), first (i. e., linear perturbation terms) and second (i. e., lowest order nonlinear perturbation) order, respectively.

If one expands Equation (8) to the second order the result can be written in the form,

$$W_k \simeq W_{k1} + W_{k2} \qquad , \tag{11}$$

where

$$W_{k1} = -\frac{1}{e}\left\{\rho_o\left[w_o + \left((v_o + v_1)^2 - v_o^2\right) w_o'\right] + \rho_1\left[w_o + 2\, v_o v_1 w_o'\right]\right\} , \tag{12}$$

and

$$W_{k2} = -\frac{1}{e}\left\{\rho_o\left[2\, v_o v_2 w_o' + 2 v_o^2\, v_1^2\, w_o''\right] + \rho_2 w_o\right\} \quad , \tag{13}$$

where $w_o = w(v_o)$, $w_o' = dw_o/d(v_o^2)$, and $w_o'' = d^2 w_o/d(v_o^2)^2$. The reason for choosing W_{k1} and W_{k2} in this particular manner is discussed later.

Similarly one can expand Equation (9) to the second order and write the result in the form,

$$P_{kz} \simeq P_{k1} + P_{k2} \qquad , \tag{14}$$

where

$$P_{k1} = -\frac{1}{e}\left\{(i_o + i_1)(w_o + 2v_o v_1 w_o')\right\} \quad , \tag{15}$$

and

$$P_{k2} = -\frac{1}{e}\left\{i_o(v_1^2\, w_o' + 2v_o v_2 w_o' + 2v_o^2\, v_1^2\, w_o'') + i_2 w_o\right\} \quad . \tag{16}$$

The second-order expansion of the product $i_z E_z$ can be written in the form,

$$i_z E_z = -\frac{i_z}{e}\frac{dw}{dz} \simeq (i_z E_z)_1 + (i_z E_z)_2 \quad , \tag{17}$$

where

$$(i_z E_z)_1 = -\frac{1}{e}\left[(i_o + i_1)\, D_z(w_o + 2v_o v_1 w_o')\right] \quad , \tag{18}$$

and

$$(i_z E_z)_2 = -\frac{1}{e}\left\{i_o\left[D_z(v_1^2\, w_o' + 2v_o v_2 w_o' + 2v_o^2\, v_1^2\, w_o'') - \frac{w_o'}{v_o}\frac{\partial v_1^2}{\partial t}\right] + i_2 D_z w_o\right\} \quad ; \tag{19}$$

Here D_z denotes the linearized operator d/dz, i.e.,

$$D_z = \frac{\partial}{\partial z} + \frac{1}{v_o}\frac{\partial}{\partial t} \quad .$$

Now, from the linearized space-charge-wave theory one can in principle calculate $E_o + E_1$, $i_o + i_1$, $v_o + v_1$, and $\rho_o + \rho_1$ for any given space-charge-wave device. We observe that Equation (18) is the product $(i_o + i_1)(E_o + E_1)$. Furthermore, one easily shows by the use of Equations (12), (15), (18), and $\partial i_1/\partial z = -\partial\rho_1/\partial t$ that the energy conservation relation,

$$(i_z E_z)_1 = \frac{\partial P_{k1}}{\partial z} + \frac{\partial W_{k1}}{\partial t} \quad , \tag{20}$$

is fulfilled for our choice of P_{k1} and W_{k1}. It is clear from Equations (12) and (15) that we do not need to go beyond the linearized space-charge-wave theory in order to express P_{k1} and W_{k1}. Nevertheless, since the energy conservation relation is satisfied, we can define P_{k1} as the kinetic power flow and W_{k1} as the kinetic energy density associated with the linearized space-charge waves.

By the use of Equations (13), (16), (19), and $\partial i_2/\partial z = \partial \rho_2/\partial t$, one can easily show that the energy conservation relation,

$$(i_z E_z)_2 = \frac{\partial P_{k2}}{\partial z} + \frac{\partial W_{k2}}{\partial t} \quad , \tag{21}$$

holds for P_{k2} and W_{k2}. These quantities, which are of the second order, cannot be calculated without knowing the lowest-order nonlinear space-charge-wave terms i_2, v_2 and ρ_2. Thus we define P_{k2} and W_{k2} as the second-order power flow and energy density respectively associated with the lowest-order nonlinear space-charge waves.

It should be pointed out that P_{k1} and W_{k1} contain terms of the zeroth, first, and second order, whereas P_{k2} and W_{k2} contain terms of the second order only. It follows that P_{k1} and W_{k2} include all power-flow and energy-density terms respectively of the zeroth and first order, whereas they only account for that part of the second-order terms which is to be associated with the linearized space-charge waves. The remaining second-order terms are described by P_{k2} and W_{k2}, respectively.

Thus Equations (13) and (15) do not enable one to deal with energy and power to the full second-order accuracy unless $P_{k2} = 0 = W_{k2}$, which is usually not the case. This circumstance, however, is of minor importance, since one is seldom interested in accurate power-flow calculations, whereas the energy conservation principle that applies to W_{k1} and P_{k1} is an extremely useful tool in many basic investigations.

IV. THE RELATIVISTIC EXPRESSIONS FOR P_{kl} AND W_{kl} IN NONHOMOGENEOUS BEAMS

In the relativistic case the energy of the electron is expressed by the well-known relation

$$w(v_z) = m_o c^2 \left[\left(1 - \frac{v_z^2}{c^2}\right)^{-\frac{1}{2}} - 1 \right] \quad , \tag{22}$$

where m_o is the mass of the electron at rest and c is the velocity of light in a vacuum.

By using Equation (22), one obtains from Equation (13),

$$\boxed{W_{kl} = -\frac{m_o}{e} \left\{ \rho_o \left[c^2 \left(1 - \frac{v_o^2}{c^2}\right)^{-\frac{1}{2}} - c^2 + \frac{1}{2}\left((v_o + v_1)^2 - v_o^2\right)\left(1 - \frac{v_o^2}{c^2}\right)^{-\frac{3}{2}} \right. \right. \\ \left. \left. + \rho_1 \left[c^2 \left(1 - \frac{v_o^2}{c^2}\right)^{-\frac{1}{2}} - c^2 + v_o v_1 \left(1 - \frac{v_o^2}{c^2}\right)^{-\frac{3}{2}} \right] \right\} ,} \tag{23}$$

and from Equation (15),

$$\boxed{P_{kl} = \frac{m_o}{e} (i_o + i_1) \left[c^2 \left(1 - \frac{v_o^2}{c^2}\right)^{-\frac{1}{2}} - c^2 + v_o v_1 \left(1 - \frac{v_o^2}{c^2}\right)^{-\frac{3}{2}} \right] .} \tag{24}$$

In expressions (23) and (24), v_o is an arbitrary function of z while $\rho_o(z)$ is fixed by the relation $v_o(z)\,\rho_o(z) = i_o$ = constant. The wave quantities i_1, ρ_1, and v_1 are functions of z and t to be determined from the linearized space-charge-wave theory. If $v_o^2 << c^2$, one finds that Equations (23) and (24) reduce to Equations (14 and (13) respectively of Louisell and Pierce,[3] which shows that these authors' expressions are perfectly valid for the nonhomogeneous ($v_o \neq$ constant) case.

Finally, we express the time-average kinetic a-c power flow from Equation (24),

$$\boxed{\left\langle P_{k1} \right\rangle_{ac} = -\frac{m_o}{e} v_o \left(1 - \frac{v_o^2}{c^2}\right)^{-\frac{3}{2}} \frac{1}{2} \mathrm{Re}(i_1 v_1^*)} \quad . \quad (25)$$

It should be pointed out that the important Equations (23), (24), and (25) are valid regardless of how v_o, ρ_o, and $i_o (= \rho_o v_o)$ vary with the transverse co-ordinates.

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AN INTRODUCTORY RELATIVISTIC STUDY OF THE LLEWELLYN ELECTRONIC GAP

Sven Olving

School of Electrical Engineering
CORNELL UNIVERSITY
Ithaca, New York

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ABSTRACT

The relativistic linearized electronic "telegrapher's equation" of the Llewellyn parallel-plane gap is deduced. General expressions for the a-c quantities, such as the electric field, the electron velocity, the convection current density, and the gap impedance are worked out.

I. INTRODUCTION

Suppose we have two parallel-plane electrodes (two grids, or a cathode and an anode) between which a single-velocity electron beam propagates perpendicularly to the electrodes (the positive z-direction, Figure 1). The beam and the electrodes are assumed to be sufficiently extended in the transverse direction so that fringe effects can be ignored. The d-c voltage between the electrodes is such that no electrons are reflected in the gap. There is an a-c voltage between the electrodes, and the gap is loaded by an external a-c circuit connected across the gap.

The a-c behavior of such a system was studied in detail by Llewellyn[1,2] more than twenty years ago. Llewellyn's theoretical work has been of enormous importance in connection with the understanding of high-frequency phenomena in various electron devices, such as diodes and multigrid electron tubes, klystron gaps, etc. Llewellyn's classical analysis was based on the so-called ballistic approach and involved a tedious integration procedure. Recently Rydbeck[3] has published his early (about 1953) "telegraph equation" studies of the axially inhomogeneous ionized stream. In these studies Rydbeck deals incidentally with the Llewellyn gap. His approach is based on the space-charge-wave concept, first introduced by Hahn,[4] and while his results are identical to those of Llewellyn, his approach is considerably more efficient.

One of the dominant trends in modern microwave electronics is the development of extremely high-power electron devices. Continuous-wave klystrons for 1-Mw, S-band power are now under development. Such tubes utilize relativistic electron beams (up to 300 kev). It is therefore desirable to extend the parallel-plane gap analysis to the relativistic velocity region. The purpose of this report is to present an introductory relativistic linearized gap study based on the space-charge wave approach. Explicit relativistic expressions of general nature for the various a-c quantities have been worked out. The studies will continue in an attempt to apply the general results on specific technical problems such as the space-charge-limited high-frequency diode with a relativistic anode voltage, the relativistic klystron gap, etc.

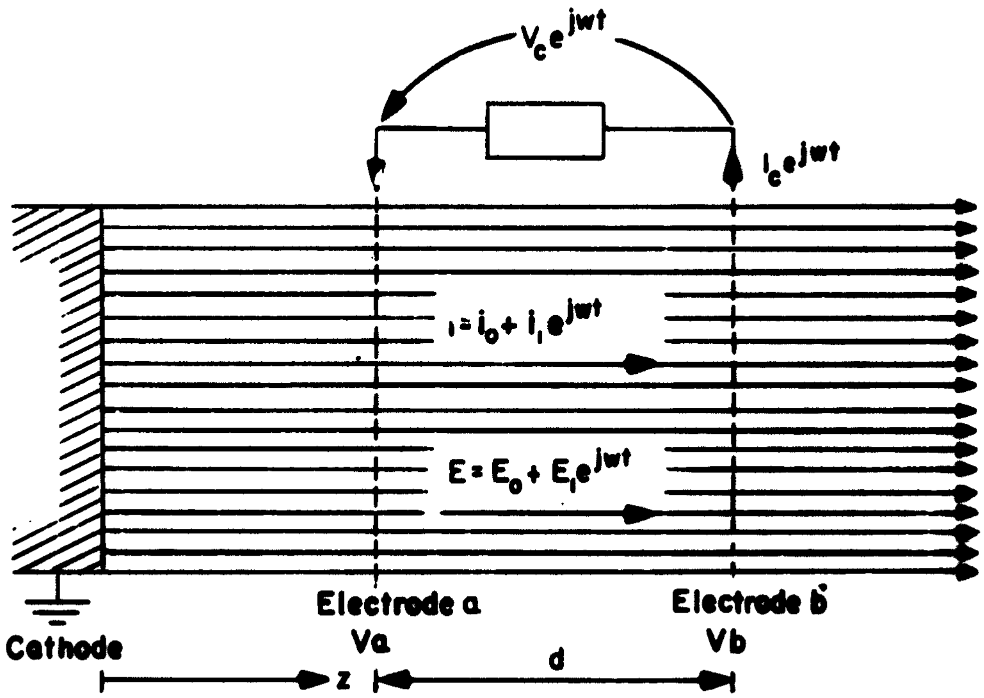

Figure 1. Llewellyn Electronic Gap.

II. THE LINEARIZED RELATIVISTIC WAVE EQUATION

To begin with, let us introduce the following fundamental notations:

ω = angular signal frequency

t = time

z = axial co-ordinate, position of the electron

$-e/m_o$ = electronic charge-to-mass ratio (at rest)

ϵ_o = permeability of vacuum

c = velocity of light in vacuum

$v(z, t)$ = $v_o(z) + v_1(z)\, e^{j\omega t}$ = total electron velocity

$v_o(z)$ = d-c velocity

$v_1(z)$ = amplitude of a-c velocity

$\rho(z, t)$ = $\rho_o(z) + \rho_1(z)\, e^{j\omega t}$ = electronic charge density

$\rho_o(z)$ = d-c charge density

$\rho_1(z)$ = amplitude of a-c charge density

$i(z, t)$ = $i_o + i_1(z)\, e^{j\omega t}$ = electronic current density

$i_o = \rho_o v_o$ = d-c current density (constant) in the positive z-direction

$i_1(z) = v_o\rho_1 + v_1\rho_o$ = amplitude of a-c current density

$E(z, t) = E_o(z) + E_1(z)\, e^{j\omega t}$ = total axial electric field in the positive z-direction

$E_o(z)$ = d-c field

$E_1(z)$ = amplitude of a-c field

A. The Relativistic Equation of Motion

$$\frac{d}{dt} \frac{v}{\left(1 - \frac{v^2}{c^2}\right)^{1/2}} = - \frac{e}{m_o} E \tag{1}$$

Remembering that $dz/dt = v$, we can write

$$\frac{d}{dz} \left[\frac{c^2}{\left(1 - \frac{v^2}{c^2}\right)^{1/2}} \right] = - \frac{e}{m_o} E \quad . \tag{2}$$

Now, if $|v_1| << c$, then

$$\frac{c^2}{\left(1 - \frac{v^2}{c^2}\right)^{1/2}} \simeq \frac{c^2}{\left(1 - \frac{v_o^2}{c^2}\right)^{1/2}} + v_1 e^{j\omega t} \frac{d}{dv_o} \left[\frac{c^2}{\left(1 - \frac{v_o^2}{c^2}\right)^{1/2}} \right] =$$

$$= \frac{c^2}{\left(1 - \frac{v_o^2}{c^2}\right)^{1/2}} + \frac{v_o v_1 e^{j\omega t}}{\left(1 - \frac{v_o^2}{c^2}\right)^{3/2}} \quad . \tag{3}$$

By the use of Equation (3) we can separate Equation (2) into its d-c and a-c parts:

$$\frac{d}{dz} \left[\frac{c^2}{\left(1 - \frac{v_o^2}{c^2}\right)^{1/2}} \right] = - \frac{e}{m_o} E_o \quad , \tag{4}$$

$$\frac{d}{dz} \left[\frac{v_o v_1 e^{j\omega t}}{\left(1 - \frac{v_o^2}{c^2}\right)^{3/2}} \right] = \frac{e}{m_o} E_1 d^{j\omega t} \quad . \tag{5}$$

A. **The Relativistic Equation of Motion**

$$\frac{d}{dt}\;\frac{v}{\left(1-\frac{v^2}{c^2}\right)^{1/2}} = -\frac{e}{m_o}\,E \tag{1}$$

Remembering that $dz/dt = v$, we can write

$$\frac{d}{dz}\left[\frac{c^2}{\left(1-\frac{v^2}{c^2}\right)^{1/2}}\right] = -\frac{e}{m_o}\,E \quad . \tag{2}$$

Now, if $|v_1| \ll c$, then

$$\frac{c^2}{\left(1-\frac{v^2}{c^2}\right)^{1/2}} \simeq \frac{c^2}{\left(1-\frac{v_o^2}{c^2}\right)^{1/2}} + v_1 e^{j\omega t}\,\frac{d}{dv_o}\left[\frac{c^2}{\left(1-\frac{v_o^2}{c^2}\right)^{1/2}}\right] =$$

$$= \frac{c^2}{\left(1-\frac{v_o^2}{c^2}\right)^{1/2}} + \frac{v_o v_1 e^{j\omega t}}{\left(1-\frac{v_o^2}{c^2}\right)^{3/2}} \quad . \tag{3}$$

By the use of Equation (3) we can separate Equation (2) into its d-c and a-c parts:

$$\frac{d}{dz}\left[\frac{c^2}{\left(1-\frac{v_o^2}{c^2}\right)^{1/2}}\right] = -\frac{e}{m_o}\,E_o \quad , \tag{4}$$

$$\frac{d}{dz}\left[\frac{v_o v_1 e^{j\omega t}}{\left(1-\frac{v_o^2}{c^2}\right)^{3/2}}\right] = \frac{e}{m_o}\,E_1\,e^{j\omega t} \quad , \tag{5}$$

The a-c velocity can now be expressed from Equation (5):

$$v_1 e^{j\omega t} = - \frac{e\left(1 - \frac{v_o^2}{c^2}\right)^{3/2}}{m_o v_o} \int E_1 e^{j\omega t}\, dz \quad . \tag{6}$$

B. The Equation of Continuity

$$\frac{\partial i_1 e^{j\omega t}}{\partial z} = - \frac{\partial \rho_1 e^{j\omega t}}{\partial t} \tag{7}$$

Since

$$\frac{d}{dz} = \frac{\partial}{\partial z} + \frac{dt}{dz}\frac{\partial}{\partial t} \simeq \frac{\partial}{\partial z} + \frac{1}{v_o}\frac{\partial}{\partial t} \quad ,$$

one can write

$$\frac{di_1 e^{j\omega t}}{dz} = \frac{\partial i_1 e^{j\omega t}}{\partial z} + \frac{1}{v_o}\frac{\partial i_1 e^{j\omega t}}{\partial t} \quad . \tag{8}$$

By the use of Equation (7) and the linearized relation $i_1 = \rho_o v_1 + \rho_1 v_o$, we obtain from Equation (8)

$$\frac{di_1 e^{j\omega t}}{dz} = j\omega \frac{\rho_o}{v_o} v_1 e^{j\omega t} \quad . \tag{9}$$

Introducing the notations,

$$\beta_{p_o}^2 = \frac{-e\rho_o}{m_o \epsilon_o v_o^2} = \frac{-e i_o}{m_o \epsilon_o v_o^3} \quad ,$$

$\beta_e = \omega/v_o$, and $k_o = \omega/c$, one obtains by the use of Equations (6) and (9)

$$i_1 e^{j\omega t} = j\omega\epsilon_o \frac{\beta_{p_o}^2}{\beta_e^2} \frac{1}{\beta_e} \int \left[\beta_e^3 \left(1 - \frac{k_o^2}{\beta_e^2}\right)^{3/2} \int E_1 e^{j\omega t}\, dz \right] dz \ , \tag{10}$$

where use has been made of the fact that $\beta_{p_o}^2/\beta_e^3$ is independent of v_o (and z).

Let us now define a wave potential $\Pi(\theta)$ by the expression,

$$\Pi = j\omega\epsilon_o \frac{1}{\beta_e} \int \left[\beta_e^2 \left(1 - \frac{k_o^2}{\beta_e^2}\right)^{3/2} \int E_1 e^{j\theta} \frac{1}{\beta_e}\, d\theta \right] d\theta \ , \tag{11}$$

where we have introduced the d-c drift angle $\theta = \int \beta_e dz$. Furthermore, with $d\theta = \beta_e dz$, one gets $\omega t = \int \omega dt \simeq \int \frac{\omega}{v_o}\, dz = \int \beta_e dz = \theta$.

E_1 can be expressed from Equation (11), viz.,

$$E_1 = \frac{1}{j\omega\epsilon_o} \beta_e e^{-j\theta} \frac{d}{d\theta} \left[\frac{1}{\beta_e^2 \left(1 - \frac{k_o^2}{\beta_e^2}\right)^{3/2}} \frac{d}{d\theta} (\beta_e \Pi) \right] , \tag{12}$$

or in an alternative form,

$$E_1 = \frac{1}{j\omega\epsilon_o} e^{-j\theta} \left\{ \frac{d}{d\theta} \left[\frac{1}{\left(1 - \frac{k_o^2}{\beta_e^2}\right)^{3/2}} \frac{d\Pi}{d\theta} \right] - \Pi\beta_e \frac{d^2}{d\theta^2} \left[\frac{\frac{1}{\beta_e}}{\left(1 - \frac{k_o^2}{\beta_e^2}\right)^{1/2}} \right] \right\} . \tag{13}$$

Equations (6) and (11) yield

$$v_1 = -\frac{1}{j\omega\epsilon_o} \frac{e}{m_o\omega} e^{-j\theta} \frac{1}{\beta_e} \frac{d}{d\theta} (\beta_e \Pi) \ . \tag{14}$$

From Equations (10) an d (11), one obtains

$$i_1 = \frac{\beta_{p_o}^2}{\beta_e^2} \; e^{-j\theta}\,\Pi \quad . \tag{15}$$

C. Gauss' Law for the Electric Field

$$\frac{\partial E_o}{\partial z} = \frac{\rho_o}{\epsilon_o} \tag{16}$$

$$\frac{\partial E_1}{\partial z} = \frac{\rho_1}{\epsilon_o} \tag{17}$$

For the moment we are going to ignore Equation (16) since we want to keep v_o a completely arbitrary function of z. However, Equations (7) and (17) yield

$$\frac{\partial}{\partial z} \left(i_1 + j\omega\epsilon_o E_1\right) = 0 \tag{18}$$

if we ignore the fringe fields. Upon integration of Equation (18),one obtains

$$i_1 + j\omega\epsilon_o E_1 = i_c \quad , \tag{19}$$

where i_c is a constant. The current $I_c e^{j\omega t} = Ai_c e^{j\omega t}$ (where A is the area of the beam cross section) is, of course, the a-c conduction current in the external circuit between the electrodes. In dealing with pure space-charge waves, one assumes that the gap is open-circuited, i. e., $i_c = 0$. With respect to the nature of our problem, we have to consider the case $i_c \neq 0$.

From Equations (10) an d (11), one obtains

$$i_1 = \frac{\beta_{p_o}^2}{\beta_e^2} e^{-j\theta} \Pi \quad . \tag{15}$$

C. Gauss' Law for the Electric Field

$$\frac{\partial E_o}{\partial z} = \frac{\rho_o}{\epsilon_o} \tag{16}$$

$$\frac{\partial E_1}{\partial z} = \frac{\rho_1}{\epsilon_o} \tag{17}$$

For the moment we are going to ignore Equation (16) since we want to keep v_o a completely arbitrary function of z. However, Equations (7) and (17) yield

$$\frac{\partial}{\partial z} (i_1 + j\omega\epsilon_o E_1) = 0 \tag{18}$$

if we ignore the fringe fields. Upon integration of Equation (18), one obtains

$$i_1 + j\omega\epsilon_o E_1 = i_c \quad , \tag{19}$$

where i_c is a constant. The current $I_c e^{j\omega t} = A i_c e^{j\omega t}$ (where A is the area of the beam cross section) is, of course, the a-c conduction current in the external circuit between the electrodes. In dealing with pure space-charge waves, one assumes that the gap is open-circuited, i.e., $i_c = 0$. With respect to the nature of our problem, we have to consider the case $i_c \neq 0$.

Upon insertion of Equations (15) and (12) or (13) into Equation (19), we obtain two alternative forms of the wave equation for Π, viz.,

$$\beta_e \frac{d}{d\theta}\left[\frac{1}{\beta_e^2\left(1-\frac{k_o^2}{\beta_e^2}\right)^{3/2}} \frac{d}{d\theta}(\beta_e \Pi)\right] + \frac{\beta_{p_o}^2}{\beta_e^2}\Pi = i_c e^{j\theta} \quad , \tag{20}$$

or

$$\frac{d}{d\theta}\left[\frac{1}{\left(1-\frac{k_o^2}{\beta_e^2}\right)^{3/2}} \frac{d\Pi}{d\theta}\right] + \left[\frac{\beta_{p_o}^2}{\beta_e^2} - \beta_e \frac{d^2}{d\theta^2}\left(\frac{\frac{1}{\beta_e}}{\left(1-\frac{k_o^2}{\beta_e^2}\right)^{1/2}}\right)\right]\Pi = i_c e^{j\theta} \quad . \tag{21}$$

III. SOLUTIONS TO THE WAVE EQUATION

The wave Equations (20) and (21) are valid for arbitrary $v_o(s)$. For example, if v_o = constant (i.e., β_e = constant, β_{p_o} = constant) and if $i_c = 0$, the solutions are

$$\Pi = \exp\left[\pm j \frac{\beta_{p_o}}{\beta_e}\left(1 - \frac{k_o^2}{\beta_e^2}\right)^{3/4} \theta\right] = \exp\left[\pm j\beta_{p_o}\left(1 - \frac{v_o^2}{c^2}\right)^{3/4} s\right],$$

which is the usual slow and fast relativistic space-charge-wave pair in a homogeneous electron stream.

However, if we do not assume the presence of positive ions, we cannot choose $v_o(s)$ arbitrarily since we have to satisfy Equation (16), which can be written

$$\frac{dE_o}{d\theta} = \frac{i_o}{\omega \epsilon_o} \quad . \tag{22}$$

Now Equation (4) can easily be written in the form,

$$\omega^2 \frac{d}{d\theta}\left[\frac{\frac{1}{\beta_e}}{\left(1 - \frac{k_o^2}{\beta_e^2}\right)^{1/2}}\right] = -\frac{e}{m_o} E_o \quad . \tag{23}$$

Upon elimination of E_o between Equations (22) and (23), one obtains

$$\beta_e \frac{d^2}{d\theta^2}\left[\frac{\frac{1}{\beta_e}}{\left(1 - \frac{k_o^2}{\beta_e^2}\right)^{1/2}}\right] = \frac{\beta_{p_o}^2}{\beta_e^2} \quad , \tag{24}$$

which reduces the wave Equation (21) to the remarkably simple form

$$\boxed{\frac{d}{d\theta}\left[\frac{1}{\left(1-\frac{k_o^2}{\beta_e^2}\right)^{3/2}}\frac{d\Pi}{d\theta}\right] = i_c e^{j\theta}} \quad . \tag{25}$$

Observe that the characteristic space-charge-wave propagation constant β_{p_o} no longer appears in the wave equation.

The general solution to Equation (25) is

$$\boxed{\Pi = \int \left(C_1 - ji_c e^{j\theta}\right)\left(1-\frac{k_o^2}{\beta_e^2}\right)^{3/2} d\theta + C_2} \quad , \tag{26}$$

where the integration constants C_1 and C_2 are to be determined from the a-c boundary conditions.

We now collect our expressions for the various a-c quantities:

$$i_1 = \frac{\beta_{p_o}^2}{\beta_e^2} e^{-j\theta}\,\Pi \quad , \tag{27}$$

$$E_1 = \frac{1}{j\omega\epsilon_o}(i_c - i_1) = \frac{1}{j\omega\epsilon_o}\left(i_c - \frac{\beta_{p_o}^2}{\beta_e^2} e^{-j\theta}\,\Pi\right) \tag{28}$$

$$v_1 = -\frac{1}{j\omega\epsilon_o}\,\frac{e}{m_o\omega}\,e^{-j\theta}\left(\frac{d\Pi}{d\theta} - \Pi\,\frac{1}{\frac{1}{\beta_e}}\,\frac{d\frac{1}{\beta_e}}{d\theta}\right)$$

$$= -\frac{1}{j\omega\epsilon_o}\,\frac{e}{m_o\omega}\,e^{-j\theta}\left\{\left(C_1 - ji_c e^{j\theta}\right)\left(1-\frac{k_o^2}{\beta_e^2}\right)^{3/2}\right.$$

$$- \frac{1}{\frac{1}{\beta_e}} \frac{d\frac{1}{\beta_e}}{d\theta}\left[\int \left(C_1 - ji_c e^{j\theta}\right) \left(1 - \frac{k_o^2}{\beta_e^2}\right)^{3/2} d\theta - C_2\right]\Bigg\} \tag{29}$$

The a-c voltage amplitude across the gap becomes

$$V_c = \int_o^d E_1 dz = \int_o^{\theta_o} \frac{E_1}{\beta_e} d\theta = \frac{1}{j\omega\epsilon_o}\left(i_c \int_o^{\theta_o} \frac{d\theta}{\beta_e} - \frac{\beta_{p_o}^2}{\beta_e^3} \int_o^{\theta_o} e^{-j\theta}\, \Pi d\theta\right), \tag{30}$$

where d is the distance between the electrodes and θ_o is the corresponding d-c transit angle $\int_o^d \beta_e dz$.

The a-c gap impedance Z_c finally is easily expressed from

$$Z_c = \frac{V_c}{I_c} = \frac{V_c}{Ai_c} \quad . \tag{31}$$

Since $\int_o^{\theta_o} d\theta/\beta_e = d$, one observes from Equations (30) and (31) that the cold $(\beta_{p_o} = 0)$ gap impedance Z_{c_o} becomes as expected,

$$Z_{c_o} = \frac{1}{j\omega C_o} \quad , \tag{32}$$

where $C_o = \epsilon_o A/d$ is the cold-gap capacitance.

With Equations (26) through (31), we have reached the principal goal of the present introductory report. A continuation of the study is under way with the aim of computing the complete set of Llewellyn coefficients with the relativistic corrections taken into account at least to the first order.

It should be pointed out that for the convenience of the general reader, the usefulness of Equations (26) to (31) is demonstrated in the Appendix by applying these relations to the space-charge-limited high-frequency nonrelativistic diode.

IV. THE BASIC D-C EQUATIONS FOR THE GAP

In order to evaluate the integrals appearing in the a-c Equations (26) to (31), one has to express $\beta_e(z)$ in terms of $\theta(z)$. The purpose of this section is to deduce the necessary d-c relations.

If V_a and V_b are the d-c potentials of the two electrodes with respect to the cathode, one gets from Equation (4) the energy relation,

$$c^2 \left[\frac{1}{\left(1 - \dfrac{k_o^2}{\beta_{e_{a,b}}^2}\right)^{1/2}} - 1 \right] = \frac{e}{m_o} V_{a,b} \ , \qquad (33)$$

which can be written

$$\frac{\dfrac{1}{\beta_{e_{a,b}}}}{\left(1 - \dfrac{k_o^2}{\beta_{e_{a,b}}^2}\right)^{1/2}} = \frac{1}{k_o} \left[\frac{2eV_{a,b}}{m_o c^2} \left(1 + \frac{eV_{a,b}}{2m_o c^2}\right)\right]^{1/2} \ . \qquad (34)$$

Relation (34) expresses the boundary conditions of Equation (24), which can be integrated to obtain

$$\frac{\dfrac{1}{\beta_e}}{\left(1 - \dfrac{k_o^2}{\beta_e^2}\right)^{1/2}} = \frac{\beta_{p_o}^2}{\beta_e^3} \ \frac{\theta}{2} \ (\theta - \theta_o) + \frac{1}{k_o} \left\{ \left[\frac{2eV_b}{m_o c^2} \left(1 + \frac{eV_b}{2m_o c^2}\right)\right]^{1/2} \right.$$

$$- \left[\frac{2eV_a}{m_o c^2} \left(1 + \frac{eV_a}{2m_o c^2} \right) \right]^{1/2} \Bigg\} \frac{\theta}{\theta_o} + \frac{1}{k_o} \left[\frac{2eV_a}{m_b c^2} \left(1 + \frac{eV_a}{2m_o c^2} \right) \right]^{1/2} . \tag{35}$$

We denote the right-hand side of Equation (35) by $S(\theta)$ and express

$$\frac{1}{\beta_e} = \frac{S}{(1 + k_o^2 S^2)^{1/2}} , \tag{36}$$

$$\left(1 - \frac{k_o^2}{\beta_e^2} \right)^{1/2} = \frac{1}{(1 + k_o^2 S^2)^{1/2}} . \tag{37}$$

Since S is a known function of θ, it would now be possible to eliminate β_e from our a-c expressions by the use of the last two relations. The whole problem is thereby reduced to the evaluation of the integrals involved and to the application of the a-c boundary conditions with which we will not concern ourselves in the present report.

Finally we want to formulate an equation from which the d-c transit angle θ_o can be expressed in terms of i_o, V_a, V_b, d, and ω. One has

$$d = \int_0^{\theta_o} \frac{d\theta}{\beta_e} = \int_0^{\theta_o} \frac{S}{(1 + k_o^2 S^2)^{1/2}} \, d\theta , \tag{38}$$

which is the desired equation.

APPENDIX: THE NONRELATIVISTIC SPACE-CHARGE-LIMITED DIODE

In the case of the nonrelativistic space-charge-limited diode, Equation (24) yields

$$\frac{1}{\frac{1}{\beta_e}} \frac{d\left(\frac{1}{\beta_e}\right)}{d\theta} = \frac{\beta_{p_o}^2}{\beta_e^2} \theta \quad , \tag{A. 1}$$

and

$$1 = \frac{\beta_{p_o}^2}{\beta_e^2} \frac{\theta^2}{2} \quad . \tag{A. 2}$$

By the use of Equations (A. 1), (A. 2) and (26), one obtains from expressions (27), (28), and (29), respectively

$$i_1 = 2\left(\frac{C_1}{\theta^2} e^{-j\theta} + \frac{C_2}{\theta} e^{-j\theta} - \frac{i_c}{\theta^2}\right) \quad , \tag{A. 3}$$

$$E_1 = \frac{1}{j\omega\epsilon_o} \left(i_c - \frac{2C_1}{\theta^2} e^{-j\theta} - \frac{2C_2}{\theta} e^{-j\theta} + \frac{2i_c}{\theta^2}\right) \quad , \tag{A. 4}$$

$$v_1 = \frac{1}{j\omega\epsilon_o} \frac{e}{m_o\omega} \left(C_2 e^{-j\theta} + ji_c + \frac{2C_1}{\theta} e^{-j\theta} + \frac{2i_c}{\theta}\right) \quad . \tag{A. 5}$$

The space-charge-limited initial conditions are $E_1 \to 0$, $v_1 \to 0$ when $\theta \to 0$. By the use of the expansion,

$$e^{-j\theta} \rightarrow 1 - j\theta - (1/2)\,\theta^2 \;, \; . \; . \; . \quad ,$$

one now finds

$$C_1 = i_c \qquad C_2 = j i_c \quad . \tag{A. 6}$$

Thus E_1 can be expressed as

$$E_1 = \frac{i_c}{j\omega\epsilon_o}\left(1 - \frac{2e^{-j\theta}}{\theta^2} - j\,\frac{2e^{-j\theta}}{\theta} + \frac{2}{\theta^2}\right) \quad , \tag{A. 7}$$

and the a-c voltage across the diode becomes

$$V_c = \int_0^d E_1 dz = \int_0^{\theta_o} E_1 \frac{d\theta}{\beta_e} = \frac{i_c}{j\omega\epsilon_o}\left(\frac{\beta_{p_o}^2}{2\beta_e^3}\right)\int_0^{\theta_o}\left[\theta^2 - 2e^{-j\theta} - j2\theta e^{-j\theta} + 2\right] d\theta$$

$$= \frac{i_c}{\omega\epsilon_o}\;\frac{\beta_{p_o}^2}{\beta_e^3}\left\{\left[2(1-\cos\theta_o) - \theta_o\sin\theta_o\right]\right.$$

$$\left. - j\left[\frac{1}{6}\,\theta_o^3 + \theta_o(1+\cos\theta_o) - 2\sin\theta_o\right]\right\} \quad , \tag{A. 8}$$

where use has been made of Equation (A. 2) and the fact that $\beta_{p_o}^2/\beta_e^3$ is a constant. One easily finds

$$\frac{1}{A\omega\epsilon_o}\;\frac{\beta_{p_o}^2}{\beta_e^3}\;\frac{\theta_o^4}{12} = \frac{2}{3}\;\frac{V_o}{I_o} \equiv R_o \quad , \tag{A. 9}$$

where V_o = d-c anode voltage, I_o = total d-c electron current, R_o = the differential low-frequency impedance dV_o/dI_o, from the energy relation $m_o v_o^2/2 = e V_o$ and Equation (A. 2).

The high-frequency diode impedance can now be written

$$Z_c = \frac{V_c}{Ai_c} = R_o \frac{12}{\theta_o^4} \left\{ \left[2(1 - \cos\theta_o) - \theta_o \sin\theta_o \right] - j \left[\frac{1}{6}\theta_o^3 + \theta_o (1 + \cos\theta_o) - 2\sin\theta_o \right] \right\} , \qquad \text{(A. 10)}$$

which is the well-known expression obtained by Llewellyn[1] in an entirely different manner.

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